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MONOIDAL EXTENSIONS OF A COHEN-MACAULAY UNIQUE FACTORIZATION DOMAIN

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Abstract. Let A be a Noetherian Cohen-Macaulay domain, b, $c_1, \ldots, c_g$ an A-sequence, $J = (b, c_1, \ldots, c_g)A$, and $B = A[J/b]$. Then B is Cohen-Macaulay, there is a natural one-to-one correspondence between the sets $\mathrm{Ass}_B(B/bB)$ and $\mathrm{Ass}_A(A/J)$, and each $q \in \mathrm{Ass}_A(A/J)$ has height $g+1$. If B does not have unique factorization, then some height-one prime ideals P of B are not principal. These primes are identified in terms of J and $P \cap A$, and we consider the question of how far from principal they can be. If A is integrally closed, necessary and sufficient conditions are given for B to be integrally closed, and sufficient conditions are given for B to be a UFD or a Krull domain whose class group is torsion, finite, or finite cyclic.

It is shown that if P is a height-one prime ideal of B, then $P \cap A$ also has height one if and only if $b \notin P$ and thus $P \cap A$ has height one for all but finitely many of the height-one primes P of B. If A has unique factorization, a description is given of whether or not such a prime P is a principal prime ideal, or has a principal primary ideal, in terms of properties of $P \cap A$. A similar description is also given for the height-one prime ideals P of B with $P \cap A$ of height greater than one, if the prime factors of b satisfy a mild condition.

If A is a UFD and b is a power of a prime element, then B is a Krull domain with torsion class group if and only if J is primary and integrally closed, and if this holds, then B has finite cyclic class group. Also, if J is not primary, then for each height-one prime ideal p contained in at least one, but not all, prime divisors of J, it holds that the height-one prime $pA[1/b] \cap B$ has no principal primary ideals. This applies in particular to the Rees ring $\mathbf{R} = A[1/t, tJ]$. As an application of these results, it is shown how to construct for any finitely generated abelian group G, a monoidal transform $B = A[J/b]$ such that A is a UFD, B is Cohen-Macaulay and integrally closed, and $G \cong \mathrm{Cl}(B)$, the divisor class group of B.

1. Introduction

Suppose A is a Noetherian unique factorization domain (UFD) and $b, c \in A$ are nonzero nonunits with no common factors. The study of the simple birational extension $B := A[c/b]$ arises naturally in commutative algebra. There are many interesting examples having this form. For instance, if X, Y are indeterminates over a field F and A is the polynomial ring $F[X, Y]$, then $B := A[Y(Y-1)/X]$ is a 2-dimensional regular affine domain that is not a UFD, but $\mathrm{Spec}(B)$ is the gluing together of the two planes $\mathrm{Spec}(F[X, Y/X])$ and $\mathrm{Spec}(F[X, (Y-1)/X])$. If

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P is a prime ideal of either the polynomial ring $C_1 = F[X, Y/X]$ or the polynomial ring $C_2 = F[X, (Y-1)/X]$, then $(C_i)_P = B_{P \cap B}$. Therefore C_1 and C_2 are flat sublocalization extensions of B. Since the only units of C_i are the nonzero elements of F, C_i is not a localization of B. As we discuss below, this is possible because of the existence in B of height-one primes that have no principal primary ideals.

Our investigation of the subject of birational extensions of a Noetherian UFD is motivated by the recent paper [9] where **simple** birational extensions of A are considered. It is well known that $B = A[c/b]$ fails to be a UFD if and only if there exist in B prime ideals of height one that are not principal. A goal of [9] and also of the present paper is to identify all such primes of B and to consider how far from principal they can be. It is shown in [9, Theorem 2.8] that if b is irreducible in A and bB has more than one minimal prime, then each of the minimal primes of bB fails to be the radical of a principal ideal.

There is much that has already been done in this area. Our assumption that b and c have no common factors implies that if $(b, c)A$ is a proper ideal of A, then b, c is a regular sequence. We have found that many results can be obtained in the more general setting where $b, c_1, \ldots, c_g$, $g \geq 1$, is an A-sequence of a Cohen-Macaulay UFD and $B := A[c_1/b, \ldots, c_g/b]$. In this setting, B is an affine piece of the blowup of the ideal of the principal class $J := (b, c_1, \ldots, c_g)A$. The assumption that A is Cohen-Macaulay implies, in particular, that A is Noetherian. Several references with interesting results on the passage of A to $B = A[c_1/b, \ldots, c_g/b]$ are [14], [1], [2], [6], [7].

Let $P \in \operatorname{Spec}(B)$ be a height-one prime ideal. A goal of this paper is to identify in terms of J and $P \cap A$: (1) the height-one prime ideals of B that are principal; and, (2) the height-one prime ideals of B that have a principal primary ideal. The result [9, Theorem 2.8] mentioned above is a good example of the type of result we are interested in proving.

Assume A is Cohen-Macaulay, $b, c_1, \ldots, c_g$ is an A-sequence, $J = (b, c_1, \ldots, c_g)A$, and $B = A[J/b]$. In Proposition 2.2, we observe that B is also Cohen-Macaulay, that $b, c_1/b, \ldots, c_g/b$ is a permutable B-sequence, $bB \cap A = J$, and $B/bB \cong (A/J)[X_1, \ldots, X_g]$. Thus there exists a one-to-one correspondence between the elements of $\operatorname{Ass}_B(B/bB)$ and the elements of $\operatorname{Ass}_A(A/J)$ given by p $(\in \operatorname{Ass}_A(A/J)) = P \cap A$ with $P \in \operatorname{Ass}_B(B/bB)$ and $P = pB$. Also, each $q \in \operatorname{Ass}_A(A/J)$ has height $g+1$. If A as above is also an integrally closed domain, necessary and sufficient conditions are given in Remark 2.3 for B to be integrally closed. The main result of Section 2, Theorem 2.4, gives sufficient conditions for B to be a UFD or a Krull domain whose class group is torsion, finite, or finite cyclic.

In Section 3 we concentrate on the height-one prime ideals $P \in \operatorname{Spec}(B)$ such that $p := P \cap A$ also has height one. This happens if and only if $b \notin P$ and thus includes all but finitely many of the height-one primes of B. The main results of Section 3, summarized in Theorem 3.14 and Corollary 3.15, apply in the case where A as above is a UFD and show that $P = pB$ (a principal prime ideal) if and only if $p \not\subseteq \bigcup\{q \mid q \in \operatorname{Ass}_A(A/J)\}$. Also, if $p \subseteq \bigcup\{q \mid q \in \operatorname{Ass}_A(A/J)\}$, then P $(= pA[1/b] \cap B)$ has a principal primary ideal if and only if there exists a positive integer h and an element $x \in p \cap J^h$ such that either $b, x/b^h$ is a B-sequence or $(b, x/b^h)B = B$. For a height-one prime ideal $P \in \operatorname{Spec}(B)$ such that $P \cap A = p$ also has height one, this gives a complete description of whether or not P is a principal prime ideal, or has a principal primary ideal, in terms of properties of $P \cap A$.

In Section 4 we concentrate on the height-one prime ideals P in B such that $\mathrm{ht}(P \cap A) > 1$. It is shown that if the prime factors of b satisfy a mild condition, then P is principal (resp., has a principal primary ideal) if and only if for some prime factor b_i of b, the ideal $(b_i, c_1, \dots, c_g)A = P \cap A$ (resp., $(b_i, c_1, \dots, c_g)A$ is $(P \cap A)$-primary).

In Section 5 we consider the case where b is a power of a prime element. In this case it is shown that if A is a UFD, then B is a Krull domain with torsion class group if and only if J is primary and integrally closed, and if this holds, then B has finite cyclic class group. Also, if J is not primary, then for each height-one prime ideal p contained in at least one, but not all, prime divisors of J, it holds that the height-one prime $pA[1/b] \cap B$ has no principal primary ideals.

In Section 6 we apply the previous results to the Rees ring $\mathbf{R} = A[1/t, tJ]$. It is shown that $\mathbf{R}$ is a UFD if and only if J is prime (and then each of the rings $B_j = A[J/c_j]$ is a UFD), and that $\mathbf{R}$ is a Krull domain with torsion class group if and only if J is primary and integrally closed, and if this holds, then $\mathbf{R}$ has finite cyclic class group and each of the rings B_j is a Krull domain with a finite cyclic class group.

Finally, in Section 7 we illustrate some of the results in this paper by showing how to produce, for any finitely generated abelian group G and any integrally closed Cohen-Macaulay domain R containing a field of characteristic zero, a monoidal transform B of $R[X, Y]$ which is integrally closed and Cohen-Macaulay, and such that there is an exact sequence $0 \to G \to \mathrm{Cl}(B) \to \mathrm{Cl}(R) \to 0$.

2. A sufficient condition for B to be a Cohen-Macaulay UFD

Let A be a Cohen-Macaulay ring, let $b, c_1, \dots, c_g$ ($g \geq 1$) be an A-sequence, let $I = (c_1, \dots, c_g)A$, let $J = (b, I)A = (b, c_1, \dots, c_g)A$, let $B = A[J/b] = A[I/b] = A[c_1/b, \dots, c_g/b]$, and let K be the kernel of the A-homomorphism $A[X_1, \dots, X_g] \to B$ defined by $X_i \mapsto c_i/b$. It is shown in Proposition 2.2 that B is also Cohen-Macaulay.

It is well known that a domain R is a UFD if and only if R is a Krull domain with trivial divisor class group $\mathrm{Cl}(R)$, and the group $\mathrm{Cl}(R)$ is often taken as a measure of the failure of unique factorization of the Krull domain R. For example see [5]. Since B is Cohen-Macaulay, B satisfies Serre's condition S_2, and thus B is a Krull domain if and only if B_P is integrally closed for each height-one prime P of B. In Remark 2.3, we observe that B_P is integrally closed for all height-one primes P of B except possibly for those $P \in \mathrm{Ass}_B(B/bB)$, and the integral closedness of B_P for these remaining primes is equivalent to J being integrally closed. The main result in this section, Theorem 2.4, specifies relations between the divisor class groups $\mathrm{Cl}(A)$ and $\mathrm{Cl}(B)$. In particular, sufficient conditions are given for B to be a UFD or a Krull domain whose class group is torsion, finite, or finite cyclic. To prove these results we need several preliminary results.

The first of these is a result of E. D. Davis; it is applied in Proposition 2.2 to give the relationship between the primary decompositions of $J = JB \cap A = bB \cap A$ and $JB = bB$.

Lemma 2.1. [2, Lemma 1] *If L is an ideal of A such that $K \subseteq LA[X_1, \dots, X_g]$, then $LB \cap A = L$, $B/LB \cong (A/L)[X_1, \dots, X_g]$, and:*
(1) *If L is prime (resp., primary), then LB is prime (resp., primary).*
(2) *If $L = \bigcap\{L_i\}$ for some family of ideals $\{L_i\}$, then $LB = \bigcap\{L_iB\}$.*

(3) $(L :_A H)B = LB :_B HB$ *for any ideal* H *of* A.
(4) If p *is an isolated prime divisor of* L*, then* $l_{A_p}(A_p/LA_p) = l_{B_{pB}}(B_{pB}/LB_{pB})$*, where* $l_R(M)$ *denotes the length of the* R*-module* M.

Proposition 2.2 gives some properties of $B = A[c_1/b, \ldots, c_g/b]$ that we use in the proof of Theorem 2.4.

Proposition 2.2. *B is a Cohen-Macaulay ring, $b, c_1/b, \ldots, c_g/b$ is a permutable B-sequence, $bB \cap A = J$, and $B/bB \cong (A/J)[X_1, \ldots, X_g]$, so there exists a one-to-one correspondence between the elements of $\mathrm{Ass}_B(B/bB)$ and the elements of $\mathrm{Ass}_A(A/J)$ given by p $(\in \mathrm{Ass}_A(A/J)) = P \cap A$ with $P \in \mathrm{Ass}_B(B/bB)$ and $P = pB$. Also, each $q \in \mathrm{Ass}_A(A/J)$ has height $g + 1$.*

Proof. Since A is Cohen-Macaulay and $b, c_1, \ldots, c_g$ is an A-sequence, it is shown in [14, Theorem 2.4] that B is Cohen-Macaulay and that $b, c_1/b, \ldots, c_g/b$ is a permutable B-sequence. Also, each $q \in \mathrm{Ass}_A(A/J)$ has height $g+1$, since J is generated by an A-sequence and A is Cohen-Macaulay. Finally,

$$K = (bX_1 - c_1, \ldots, bX_g - c_g)A[X_1, \ldots, X_g],$$

by [14, Lemma 2.3], so $K \subseteq JA[X_1, \ldots, X_g]$, so the remaining conclusions follow from Lemma 2.1. □

To obtain conditions for B to be integrally closed, we use the following theorem of Lipman and Mattuck (independently), which is stated in [16, page 93]. (For this result, recall that an ideal J is **pre-normal** if all large powers of J are integrally closed, and J is **normal** if J^n is integrally closed for all positive integers n.)

Theorem. *Every monoidal transform $A[I/b]$ of an integrally closed Noetherian domain A with respect to an ideal I is integrally closed if and only if I is pre-normal. (Here, b is a nonzero element of I.)*

We also use the following theorem of Goto [6, Theorem 1.1].

Theorem. *If I is an ideal in the Noetherian ring A with $\nu_A(I) = \mathrm{ht}(I) = r$ (that is, I is of the principal class), then the following are equivalent.*
(1) I is integrally closed.
(2) I is normal.
(3) For each $p \in \mathrm{Ass}_A(A/I)$, A_p is regular and $l_{A_p}((IA_p + p^2A_p)/p^2A_p) \geq r - 1$.
When this holds, each $p \in \mathrm{Ass}_A(A/I)$ is minimal over I and I is generated by an A-sequence.

Remark 2.3. If A is an integrally closed Cohen-Macaulay domain and if J is generated by the A-sequence $b, c_1, \ldots, c_g$, then the following are equivalent:
(1) B is integrally closed.
(2) B_{qB} is integrally closed for each (height-one) prime divisor q of bB.
(3) J is integrally closed.
(4) J is pre-normal.
(5) J is normal.

Proof. B is Cohen-Macaulay, by Proposition 2.2, so the prime divisors of nonzero principal ideals have height one. Also, if $b \notin P \in \mathrm{Spec}(B)$, then $B_P = A_{P\cap A}$ (since $A[1/b] = B[1/b]$), so B_P is integrally closed (since $A_{P\cap A}$ is). It therefore follows from Nagata's definition of a Krull domain (on p. 115 of [12]) that (1) and (2) are equivalent. It is clear that (5) implies (4), and (4) implies (1) by the result of

Lipman and Mattuck quoted above. Then (1) implies (3), since B integrally closed implies that bB is integrally closed, and by Proposition 2.2, $bB \cap A = J$. Finally, by the above result of Goto, (3) implies (5). $\square$

We can now prove the main result in this section. It includes a sufficient condition for B to be a Cohen-Macaulay UFD. As mentioned in the introduction to this section, if R is a Krull domain, we let $\mathrm{Cl}(R)$ denote the divisor class group of R.

Theorem 2.4. *Assume that A is an integrally closed Cohen-Macaulay domain and that J is generated by the A-sequence $b, c_1, \dots, c_g$. Then:*
(1) *If J is integrally closed, then B is integrally closed and there is a surjective homomorphism $\varphi : \mathrm{Cl}(B) \to \mathrm{Cl}(B[1/b])$ $(= \mathrm{Cl}(A[1/b]))$ whose kernel is generated by the classes of the elements of $\mathrm{Ass}_B(B/bB)$.*
(2) *If J is integrally closed and primary, and if $\mathrm{Cl}(A)$ is torsion (resp., finite) (resp., trivial), then $\mathrm{Cl}(B)$ is torsion (resp., finite) (resp., finite cyclic).*
(3) *If J is prime and b is a product of prime elements of A, then $bB \in \mathrm{Spec}(B)$ and the divisor class groups $\mathrm{Cl}(A)$ and $\mathrm{Cl}(B)$ are isomorphic.*
(4) *If J is prime and A is a UFD, then B is a UFD.*

Proof. For (1), if J is integrally closed and is generated by an A-sequence, then B is integrally closed, by Remark 2.3(3) $\Rightarrow$ (1). The rest of (1) follows from [5, Corollary 7.2].

For (2), if J is integrally closed, then B is a Krull domain, by (1). Also, if J is primary, then bB is primary by Proposition 2.2, and thus $\mathrm{Ker}(\varphi)$ is the finite cyclic subgroup of $\mathrm{Cl}(B)$ generated by the class of $\mathrm{Rad}(bB)$. Since there is also a surjective homomorphism $\mathrm{Cl}(A) \to \mathrm{Cl}(A[1/b])$ by [5, Corollary 7.2], (2) follows.

For (3), if J is prime, then B is a Krull domain (by (1), since J is integrally closed) and bB is prime by Proposition 2.2. Therefore φ is an isomorphism. Also, since b is a product of prime elements, the canonical map $\mathrm{Cl}(A) \to \mathrm{Cl}(A[1/b])$ is also an isomorphism by Nagata's Theorem [5, Corollary 7.3]. (4) is immediate from (3). $\square$

In regard to Theorem 2.4(3), it is shown in Theorem 5.2 below that if A as in Theorem 2.4 is a UFD and b is a power of a prime element, then J is primary and integrally closed if and only if B is a Krull domain with finite cyclic divisor class group. And it is shown in Theorem 6.6 that the Rees ring $\mathbf{R} = A[1/t, tJ]$ is Krull with torsion (equivalently, finite) divisor class group if and only if J is primary and integrally closed.

The following special cases of Theorem 2.4(3) have previously appeared.

Corollary 2.5. [17, Proposition 7.6, p. 28] *If A is an integrally closed Noetherian domain, if $aA \cap bA = abA$, and if aA and $(a,b)A$ are prime ideals, then $A' = A[X]/(aX - b)$ is again integrally closed and the class groups $\mathrm{Cl}(A)$ and $\mathrm{Cl}(A')$ are canonically isomorphic.*

Corollary 2.6. [9, Corollary 2.4] *If A is a Noetherian UFD, if $aA \cap bA = abA$, and if $(a,b)A$ is a prime ideal, then $B = A[b/a]$ is a UFD.*

Concerning the conclusion that $bB \in \mathrm{Spec}(B)$ in Theorem 2.4, it is possible that bB is a prime ideal and bA is not. An example of this is given in Example 4.2.

Corollary 2.7. *Assume A is a Cohen-Macaulay UFD, that $b, c_1, \dots, c_g$ is a permutable A-sequence, and that $J = (b, c_1, \dots, c_g)A$ is a prime ideal. Then each of the rings $B_j = A[J/c_j]$ is a Cohen-Macaulay UFD and $c_j B_j \in \mathrm{Spec}(B_j)$.*

Proof. This follows immediately from Theorem 2.4(4), if $b, c_1, \dots, c_g$ is a permutable A-sequence. $\square$

It is shown in Corollary 6.5 that it is not necessary to assume that $b, c_1, \dots, c_g$ is a *permutable* A-sequence in Corollary 2.7.

Example 2.8 shows that the sufficient condition in Theorem 2.4(4) for B to be a UFD, that J is a prime ideal generated by an A-sequence, is not a necessary condition.

Example 2.8. Let

$$A = \mathbb{Z}[X, Y] \quad \text{and} \quad B = A[p/Y, (XY)/p] = A[p^2/(pY), (XY^2)/(pY)],$$

where $\mathbb{Z}$ is the ring of integers, p is a prime integer and X, Y are algebraically independent over $\mathbb{Z}$. Then B is a Cohen-Macaulay UFD, but $(pY, p^2, XY^2)A$ is not a prime ideal.

Proof. B is a UFD by two applications of Theorem 2.4, namely: $A \to C = A[p/Y] \to C[X/(p/Y)]$, and each step uses a prime ideal ($A \to C$ uses $(p, Y)A$, and $C \to B$ uses $(p/Y, X)C$). Finally, it is clear that $(pY, p^2, XY^2)A$ has the two prime divisors $(p, X)A$ and $(p, Y)A$. $\square$

An example similar to Example 2.8 is $A = F[X, Y, Z]$ and $B = A[Z/X, (XY)/Z] = A[Z^2/(XZ), (X^2Y)/(XZ)]$, where F is a field and X, Y, Z are algebraically independent over F. Then B is a Cohen-Macaulay UFD, but $(XZ, Z^2, X^2Y)A$ is not a prime ideal. In this example, B is generated over F by $X, Z/X, XY/Z$ and so is a polynomial ring in these three elements over F. Similarly the rings $B_2 = A[Z/(XY)]$, $B_3 = A[Y/X, (XZ)/Y]$, $B_4 = A[Y/(XZ)]$, $B_5 = A[X/Y, (YZ)/X]$, and $B_6 = A[X/(YZ)]$ are all polynomial rings over F and are therefore UFDs.

The following lemma is frequently used below.

Lemma 2.9. *Let $P \in \operatorname{Spec}(B)$ and let $p = P \cap A$. Then the following hold:*
(1) $\operatorname{ht}(P) \le \operatorname{ht}(p) \le \operatorname{ht}(P) + g$.
(2) *If $b \notin p$, then $\operatorname{ht}(p) = \operatorname{ht}(P)$ and $P = pA[1/b] \cap B$ (so $B_P = A_p$).*
(3) *If $\operatorname{ht}(P) = 1$, then $\operatorname{ht}(p)$ is either 1 or $g+1$, and $\operatorname{ht}(p) = g+1$ if and only if $b \in p$ if and only if $p \in \operatorname{Ass}_A(A/J)$. Moreover, if $\operatorname{ht}(p) = g+1$, then $P = pB$.*

Proof. (1) follows from the fact that A satisfies the Altitude Equality ($\operatorname{ht}(P) + \operatorname{tr.deg}_{A/p}(B/P) = \operatorname{ht}(p) + \operatorname{tr.deg}_A(B) = \operatorname{ht}(p)$, since A is Cohen-Macaulay) and $A[1/b] = B[1/b]$). (2) follows from $A[1/b] = B[1/b]$).

For (3), if $\operatorname{ht}(p) > 1$, then $b \in p$, by (2), so $J \subseteq p$ and $\operatorname{ht}(p) \ge \operatorname{ht}(J) = g+1$, hence $\operatorname{ht}(p) = g+1$, by (1). Therefore $p \in \operatorname{Ass}_A(A/J)$, so $P = pB$, by Proposition 2.2. And, conversely, if $b \in p = P \cap A$, then $J \subseteq P \cap A = p$, hence $\operatorname{ht}(p) > 1$. $\square$

3. The case where height$(P \cap A) = 1$

In this section we are interested in how close B is to being a UFD if J is not necessarily a primary ideal (as was assumed in Theorem 2.4(2)–(4)). So we are especially interested in which height-one prime ideals P of B: (1) are principal; or, (2) have a principal primary ideal. (Note that, since B is Cohen-Macaulay, by Proposition 2.2, *P has a principal primary ideal if and only if P is the radical of a principal ideal.)* The ideal bB plays a crucial role in answering (1) and (2), and it is the (height-one) prime divisors of bB that are the hardest to handle, so we delay considering them until Section 4. Our results in this section completely answer (1) and (2) above (except for the prime divisors of bB).

Many of the results in this section do not require the hypothesis that A is a UFD, so we do not assume that A is a UFD until Remark 3.8. Therefore, the running hypothesis up through Remark 3.7 is that A is a Cohen-Macaulay ring, $b, c_1, \dots, c_g$ is an A-sequence, $J = (b, I)A = (b, c_1, \dots, c_g)A$, and $B = A[J/b] = A[I/b]$.

We begin with a remark that describes the relationship between the height-one primes in A and the height-one primes in B. We say that a prime ideal P of A is **not lost** in B if there exists a prime ideal P' of B such that $P' \cap A = P$ [19, page 325].

Remark 3.1. Let P be a height-one prime ideal in B and let $p = P \cap A$. Then:
(1) $\mathrm{ht}(p) = 1$ if and only if $b \notin p$. In this case $P = pA[1/b] \cap B$ by Lemma 2.9(2).
(2) $\mathrm{ht}(p) > 1$ if and only if $\mathrm{ht}(p) = g + 1$ if and only if $p \in \mathrm{Ass}_A(A/J)$ by Lemma 2.9(3). In this case $P = pB$.
(3) Among the height-one prime ideals of A, only the (height-one) prime divisors of bA are lost in B (since $A[1/b] = B[1/b]$ and $J \subseteq bB$).

In this section we concentrate on the primes in (1) of Remark 3.1. For these primes the further property: $p \subseteq \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}$ plays an important role in our investigation of whether or not P is principal or has a principal primary ideal. We concentrate on the primes in (2) of Remark 3.1 in Section 4.

We frequently use the following lemma. (Our first use of it is in the statement of Proposition 3.3.) The lemma shows that each element in $B - J$ has a unique representation of the form x/b^h (with $x \in J^h - (J^{h+1} \cup bA)$). (Note that the elements in J cannot be written in this form.)

Lemma 3.2. *For each element $\beta \in B - J$ there exists a unique nonnegative integer h and a unique element $x \in J^h - (J^{h+1} \cup bA)$ such that $\beta = x/b^h$. (If $\beta \in B - A$, then $h > 0$.)*

Proof. It is clear that every element in $B - J$ may be written $y_k(b, c_1, \dots, c_g)/b^k$ for all large integers k, where $y_k(X_0, X_1, \dots, X_g) \in A[X_0, X_1, \dots, X_g]$ is a form of degree k. It then follows from a result of Rees [10, Theorem 16.2], that

$$y_k(b, c_1, \dots, c_g) \in J^k - J^{k+1}.$$

Therefore assume that $\beta \in B - J$ and let $\beta = y_k/b^k$ with $y_k \in J^k - J^{k+1}$ and k a large integer. If $y_k \in bA$, then let i be the positive integer such that $y_k \in b^iA - b^{i+1}A$, so $y_k = xb^i$ for some $x \in A$ (and note that $i \le k$). It then follows that $x \in J^k :_A b^iA = J^{k-i}$, by [14, Corollary 3.7], so $\beta = x/b^{k-i}$ with $x \in J^{k-i}$, and it is readily checked that $x \notin (J^{k-i+1} \cup bA)$. And it is now straightforward to check that the integer h and the element x are unique. Finally, it is clear that $h > 0$ if $\beta \notin A$. ☐

In Proposition 3.3 several characterizations are given for an element $\beta \in B$ to have the property that either b, β is a B-sequence or $(b, \beta)B = B$. This result plays an essential role in answering (1) and (2) in the first paragraph of this section. Since $JB = bB$, we restrict attention in this proposition to the elements in $B - J$. Also, in Proposition 3.3 we use the **Rees ring $\mathbf{R}(A, J)$ of A with respect to J**, so $\mathbf{R}(A, J)$ is the graded subring $A[u, tJ]$ of $A[u, t]$, where t is an indeterminate and $u = 1/t$.

Proposition 3.3. *Let β be a nonzero nonunit in $B - J$, and let h be the nonnegative integer such that $\beta = x/b^h$, where $x \in J^h - (J^{h+1} \cup bA)$. Also, let $\mathbf{R} = \mathbf{R}(A, J) = A[u, tb, tc_1, \ldots, tc_g]$. Then the following are equivalent:*
(1) *$\beta B :_B bB = \beta B$.*
(2) *Either b, β is a B-sequence or $(b, \beta)B = B$.*
(3) *$\beta B[1/b] \cap B = \beta B$.*
(4) *Either $u, t^h x$ is an $\mathbf{R}$-sequence or $(u, t^h x)\mathbf{R} = \mathbf{R}$ (in which case $h = 0$ and $(J, x)A = A$).*
(5) *$x \in J^h - \bigcup\{qJ^h \mid q \in \operatorname{Ass}_A(A/J)\}$.*
(6) *$x + J^{h+1}$ is a regular nonunit in the form ring $\mathbf{F}(A, J)$ of A with respect to J.*
(7) *For all nonnegative integers e and for all $y \in J^e - J^{e+1}$ it holds that $xy \in J^{h+e} - J^{h+e+1}$.*
Assume that (1)–(7) hold. If xA is a prime (resp., primary) ideal, then βB is a prime (resp., primary) ideal, and the converse holds if $xA :_A bA = xA$.

Proof. It is clear that (1) $\Leftrightarrow$ (2), since $\beta B :_B bB = \beta B$ if and only if $bB :_B \beta B = bB$.

Also, for each ideal H in B it holds that $HB[1/b] \cap B = H :_B b^k B$ for all large integers k (since $B[1/b] = B_S$ with $S = \{b^n;\, n \geq 0\}$), so it follows that (1) $\Leftrightarrow$ (3).

Next, it is shown in [14, Theorem 3.1] that the Rees ring $\mathbf{R} = \mathbf{R}(A, J)$ is Cohen-Macaulay and that $u, tb, tc_1, \ldots, tc_g$ is an $\mathbf{R}$-sequence.

Let $\mathbf{B} = \mathbf{R}[1/(tb)]$, so $B = A[J/b] \subset A[u, tJ, 1/(tb)] = \mathbf{B}$, since $c_j/b = tc_j/(tb)$. Also, $u = b/(tb) \in B[tb, 1/(tb)]$, so $\mathbf{B} = B[tb, 1/(tb)]$ is a localization of the simple transcendental extension ring $B[tb]$ of B and $u\mathbf{B} = b\mathbf{B}$. Therefore it follows that $b, x/b^h$ is a B-sequence or $(b, x/b^h)B = B$ if and only if $u, t^h x$ is a $\mathbf{B}$-sequence or $(u, t^h x)\mathbf{B} = \mathbf{B}$. Further, if $u, t^h x$ is a $\mathbf{B}$-sequence, then $u, t^h x$ is an $\mathbf{R}$-sequence (since if $u, t^h x$ is not an $\mathbf{R}$-sequence, then $u\mathbf{R} :_{\mathbf{R}} t^h x\mathbf{R} \neq u\mathbf{R}$, so $t^h x \in p$ for some prime divisor p of $u\mathbf{R}$ (and height$(p) = 1$, since $\mathbf{R}$ is Cohen-Macaulay and u is a regular nonunit), so $tb \in p$ (since $u, t^h x$ is a $\mathbf{B}$-sequence), and this contradicts the fact that u, tb is an $\mathbf{R}$-sequence).

On the other hand, assume that $(u, t^h x)\mathbf{B} = \mathbf{B}$. If $(u, t^h x)\mathbf{R} = \mathbf{R}$, then $A = (u, t^h x)\mathbf{R} \cap A = (J, x)A$, so (4) holds (so (2) $\Rightarrow$ (4), as desired). Therefore it may be assumed that $(u, t^h x)\mathbf{R} \neq \mathbf{R}$, so it must be shown that $u, t^h x$ is an $\mathbf{R}$-sequence. For this, if $u, t^h x$ is not an $\mathbf{R}$-sequence, then $t^h x$ is in some (height-one) prime divisor p of $u\mathbf{R}$. But, since $(u, t^h x)\mathbf{B} = \mathbf{B}$, it follows that $tb \in p$, and this contradicts the fact that u, tb is a $\mathbf{R}$-sequence. Therefore the supposition that $u, t^h x$ is not an $\mathbf{R}$-sequence leads to a contradiction, so (2) $\Rightarrow$ (4). (For the parenthetical statement in (4), note that if $h > 0$, then $x \in J^h$ (by the definition of $\mathbf{R}$), so $(u, t^h x)\mathbf{R} \cap A \subseteq J$, hence $(u, t^h x)\mathbf{R} \neq \mathbf{R}$.)

To see that (4) $\Rightarrow$ (2), it is clear that if $(u, t^h x)\mathbf{R} = \mathbf{R}$, then $(u, t^h x)\mathbf{B} = \mathbf{B}$ (so $(b, x/b^h)B = B$ (since $\mathbf{B} = B[tb, 1/(tb)]$ and $u\mathbf{B} = b\mathbf{B}$), so (4) $\Rightarrow$ (2), as desired). Therefore it may be assumed that $(u, t^h x)\mathbf{R} \neq \mathbf{R}$, so $u, t^h x$ is an $\mathbf{R}$-sequence. Then either $u, t^h x$ is a $\mathbf{B}$-sequence or $(u, t^h x)\mathbf{B} = \mathbf{B}$. It then follows from $\mathbf{B} = B[tb, 1/(tb)]$ (and $u\mathbf{B} = b\mathbf{B}$) that either $b, x/b^h$ is a B-sequence or $(b, x/b^h)B = B$, so (4) $\Rightarrow$ (2).

It is shown in [15] that $\mathbf{F}(A, J) \cong \mathbf{R}/u\mathbf{R}$, so (4) $\Leftrightarrow$ (6), since $x + J^{h+1}$ corresponds (under the isomorphism) to $t^h x + u\mathbf{R}$.

Since $\mathbf{F}(A, J) = \bigoplus_{i=0}^{\infty} J^i/J^{i+1}$, it is readily seen that (6) $\Leftrightarrow$ (7).

To complete the proof of the equivalence of (1)–(7) it must be shown that (5) is equivalent to the other statements. For this, it is clear that $\mathrm{Ass}_{A[u]}(A[u]/(u,J)A[u]) = \{(u,q)A[u] \mid q \in \mathrm{Ass}_A(A/J)\}$, and it follows [14, Lemma 2.3] that if $L = \mathrm{Ker}((A[u])[X_0, X_1, \dots, X_g] \to \mathbf{R})$, then

$$L = (uX_0 - b, uX_1 - c_1, \dots, uX_g - c_g)A[u, X_0, X_1, \dots, X_g].$$

Therefore $L \subseteq (u,J)A[u, X_0, X_1, \dots, X_g]$, and so it follows (since $J\mathbf{R} \subseteq u\mathbf{R}$) that $\mathrm{Ass}_{\mathbf{R}}(\mathbf{R}/u\mathbf{R}) = \{(u,q)\mathbf{R} \mid q \in \mathrm{Ass}_A(A/J)\}$. Let $\mathrm{Ass}_A(A/J) = \{q_1, \dots, q_n\}$. Then it follows that $u, t^h x$ is an $\mathbf{R}$-sequence if and only if $t^h x \notin \bigcup\{(u,q_i)\mathbf{R} \mid i = 1, \dots, n\}$ if and only if $x \notin ((u,q_i)\mathbf{R})_{[h]} = \{a \in A \mid at^h \in (u,q_i)\mathbf{R}\}$ for $i = 1, \dots, n$. It is readily checked that $((u,q_i)\mathbf{R})_{[h]} = J^{h+1} + q_i J^h = q_i J^h$. It therefore follows that (4) $\Leftrightarrow$ (5).

Finally, assume that (1)–(7) hold. If xA is a prime (resp., primary) ideal, then $xA[1/b] \cap B$ is a prime (resp., primary) ideal. Therefore $\beta B = \beta B[1/b] \cap B$ (by (3)) $= xB[1/b] \cap B = xA[1/b] \cap B$ is a prime (resp., primary) ideal.

For the converse, assume that βB is a prime (resp., primary) ideal and that $xA :_A bA = xA$. Then $xA[1/b] = xB[1/b] = \beta B[1/b]$ is a prime (resp., primary) ideal ($\beta B[1/b] \neq B[1/b]$, by (3)), so for all large integers n it holds that $xA = xA :_A b^n A = xA[1/b] \cap A$ is a prime (resp., primary) ideal. □

Concerning Proposition 3.3, recall that if H is an ideal in a ring R and if h is a positive integer, then an element $x \in H^h$ is a **superficial element of degree** h for H in case there exists a nonnegative integer c such that $(H^{n+h} : xR) \cap H^c = H^n$ for all integers $n \geq c$. The usual way of finding a superficial element of degree h for H is to pick an element $\overline{x}$ of degree h in the form ring $\mathbf{F} = \mathbf{F}(R,H)$ of R with respect to H that is not in any prime divisor of zero that contains $\overline{H}$; equivalently (since $\mathbf{F} = \mathbf{R}(R,H)/u\mathbf{R}(R,H)$), $t^h x$ is not in any prime divisor of $u\mathbf{R}(R,H)$ that contains $tH\mathbf{R}(R,H)$. Since A is Cohen-Macaulay and J is generated by an A-sequence in Proposition 3.3, $u\mathbf{R}(A,J)$ has no prime divisor that contains $tJ\mathbf{R}(A,J)$, so, if $h > 0$, then the element $t^h x$ is a superficial element of degree h for J. A good reference for this is [18].

The next result is an immediate corollary of Proposition 3.3.

Proposition 3.4. *If $b, c_1, \dots, c_g$ is a permutable A-sequence, then for $j = 1, \dots, g$, $c_j A$ is a prime (resp., primary) ideal if and only if $(c_j/b)B$ is a prime (resp., primary) ideal.*

Proof. Since b, c_j is an A-sequence (by hypothesis) and $b, c_j/b$ is a B-sequence (by Proposition 2.2), this follows immediately from the last statement in Proposition 3.3. □

In Proposition 3.11 and Corollary 3.12 a considerable generalization of Proposition 3.4 is given.

Remark 3.5. **(1)** If $b, c_1, \dots, c_g$ is a permutable A-sequence, then $B/(c_j/b)B$ and A/c_jA have the same total quotient ring, so there exists a one-to-one correspondence between the ideals $p \in \mathrm{Ass}_A(A/c_jA)$ and $P \in \mathrm{Ass}_B(B/(c_j/b)B)$ given by $p = P \cap A$ and $P = pA[1/b] \cap B$.
(2) $B/(c_1/b, \dots, c_g/b)B = A/I$, so there exists a one-to-one correspondence between the ideals $p \in \mathrm{Ass}_A(A/I)$ and $P \in \mathrm{Ass}_B(B/(c_1/b, \dots, c_g/b)B)$ given by $p = P \cap A$ and $P = pA[1/b] \cap B$.

Proof. For (1), note that each element in B may be written in the form β/b^n for all large integers n with $\beta \in J^n$. So for all large integers n it holds that $(c_j/b)B \cap A = c_jJ^n :_A b^{n+1}A \subseteq c_jA :_A b^{n+1}A = c_jA$, since c_j, b is an A-sequence, so $(c_j/b)B \cap A \subseteq c_jA$. Also $c_j \in c_jJ^n :_A b^{n+1}A$, since $b \in J$, so $c_jA \subseteq (c_j/b)B \cap A$. Therefore $(c_j/b)B \cap A = c_jA$ for $j = 1, \dots, g$. Further, $c_jA[1/b] = (c_j/b)B[1/b]$ and $(c_j/b)B[1/b] \cap B = (c_j/b)B :_B b^nB$ for all large integers n, and $(c_j/b)B :_B b^nB = (c_j/b)B$, since b, c_j/b is a B-sequence (since $b, c_1/b, \dots, c_g/b$ is a permutable B-sequence, by Proposition 2.2). Therefore $A/c_jA \subseteq B/(c_j/b)B \subseteq A[1/b]/c_jA[1/b]$. Moreover, $A[1/b]/c_jA[1/b] = (A/c_jA)[1/(b + c_jA)]$, since b, c_j is an A-sequence, so it follows that $B/(c_j/b)B$ and $A/(c_jA)$ have the same total quotient ring for $j = 1, \dots, g$, and (1) follows from this.

(2) follows similarly since

$$B/(c_1/b, \dots, c_g/b)B = A[X_1, \dots, X_g]/(K, X_1, \dots, X_g)A[X],$$

where

$$K = Ker(A[X_1, \dots, X_g] \to B)$$

(and $K = (bX_1 - c_1, \dots, bX_g - c_g)A[X_1, \dots, X_g]$, by [14, Lemma 2.3]). So

$$(K, X_1, \dots, X_g)A[X_1, \dots, X_g] = (X_1, \dots, X_g, I)A[X_1, \dots, X_g],$$

and hence $B/(c_1/b, \dots, c_g/b)B = A/I$. □

In the next result, if $p = \pi A$ with $\pi \notin J$, then Proposition 3.3 could be used to give a short proof of the result. However, π may be in J, so we give an alternate proof that does not use Proposition 3.3. This result characterizes when a principal nonzero prime ideal p in A (with $b \notin p$) extends in B to a prime ideal.

Proposition 3.6. *Let p be a height-one principal prime ideal in A such that $b \notin p$. Then the following are equivalent:*
(1) pB is a (principal) prime ideal.
(2) $p \not\subseteq \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}$.
(3) $pB :_B bB = pB$.
(4) $pB = pA[1/b] \cap B$.

Proof. Let $S = \{b^n; n \geq 0\}$. Then $B[1/b] = B_S$, so $(pB)B[1/b] \cap B = pB :_B b^nB$ for all large integers n. Also, $B[1/b] = A[1/b]$, so $(pB)B[1/b] = pA[1/b]$, so $pB :_B b^nB = (pB)B[1/b] \cap B = pA[1/b] \cap B$. Therefore $pB = pB :_B b^nB$ if and only if $pB = pA[1/b] \cap B$, hence (3) $\Leftrightarrow$ (4).

Since $b \notin p$, $pA[1/b] \cap B$ is a height-one prime ideal that contracts in A to p, so (4) $\Rightarrow$ (1). Also, $p \subseteq pB \cap A \subseteq pB[1/b] \cap A = pA[1/b] \cap A = p$ (since $b \notin p$), so $b \notin pB$. Therefore if pB is prime, then $pB = pB[1/b] \cap B$ and $pB[1/b] \cap B = pA[1/b] \cap B$, so (1) $\Rightarrow$ (4).

Now assume that $p \subseteq q \in Ass_A(A/J)$. Then $pB \subseteq qB$, and qB is a height-one prime ideal that contracts in A to q, by Proposition 2.2. Also, $p = pB \cap A$ (as noted in the preceding paragraph) $\subseteq qB \cap A = q$, so pB is properly contained in the height-one prime ideal qB, so pB is not prime. It follows that (1) $\Rightarrow$ (2).

Finally, if pB is not prime, then $pB \neq pB :_B bB$ (by (1) $\Leftrightarrow$ (3)), hence b is in some prime divisor Q of pB. However, p is principal and B is Cohen-Macaulay (by Proposition 2.2), so $\mathrm{ht}(Q) = 1$. Therefore $Q \in \mathrm{Ass}_B(B/bB)$, so $Q \cap A \in Ass_A(A/J)$ (by Proposition 2.2), and $p = pB \cap A \subseteq Q \cap A$. It follows that (2) $\Rightarrow$ (1). □

Remark 3.7. Using Proposition 3.6 and a theorem of Nagata [17, Theorem I.6.3], the questions we are considering about the structure of the height-one prime ideals of B may be reduced to the case where A is semilocal with $\{q \mid q \in \mathrm{Ass}(A/J)\}$ as the set of maximal ideals of A.

We now add the hypothesis that A is a UFD, so *from here through the end of* §*6,* A is a (Noetherian) Cohen-Macaulay UFD, $b, c_1, \ldots, c_g$ $(g \geq 1)$ is an A-sequence, $I = (c_1, \ldots, c_g)A$, $J = (b, c_1, \ldots, c_g)A = (b, I)A$, and $B = A[c_1/b, \ldots, c_g/b]$. Also, let $b = {b_1}^{a_1} \cdots {b_d}^{a_d}$, where the b_i are non-associate prime elements in A and the a_i are positive integers.

Remark 3.8. The height-one prime ideals p in A such that $pB \cap A \neq p$ are precisely the prime ideals $b_1A, \ldots, b_dA$.

Proof. This follows immediately from Lemma 2.9(2) (and the fact that if $b_i \in Q \in \mathrm{Spec}(B)$, then $J = bB \cap A \subseteq Q \cap A$ and $\mathrm{ht}(J) > 1$). □

For a nonzero nonunit $x \in A$, Proposition 3.9 considers the situation where xB is a primary ideal.

Proposition 3.9. *Assume that x is a nonzero nonunit in A such that xB is a primary ideal. Let $P = \mathrm{Rad}(xB)$ and let $p = P \cap A$.*
(1) *If $\mathrm{ht}(p) > 1$, then $b \in \mathrm{Rad}(xA)$, $p \in \mathrm{Ass}_A(A/J)$, and $\mathrm{Rad}(xB) = pB$.*
(2) *If $\mathrm{ht}(p) = 1$, then $xA :_A b^nA$ is p-primary for all large integers n, $xB = (xA :_A b^nA)B$ for all nonnegative integers n, $P = pB$, and*

$$p \not\subseteq \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}.$$

Proof. If $\mathrm{ht}(p) > 1$, then it follows from Lemma 2.9(3) that $b \in p \in \mathrm{Ass}_A(A/J)$ and that $P = pB$. Also, if $q \in \mathrm{Ass}_A(A/xA)$ and if $b \notin q$, then $Q = qA[1/b] \cap B$ is a height-one prime divisor of xB (hence xB is Q-primary, so $Q = P$) and $Q \cap A = q \neq p = P \cap A$ (since $\mathrm{ht}(q) = 1 < \mathrm{ht}(p)$), and this is a contradiction. Therefore it follows that $b \in \mathrm{Rad}(xA)$. So (1) holds.

If $\mathrm{ht}(p) = 1$, then $b \notin P$ (else $J \subseteq p = P \cap A$ and this implies the contradiction that $\mathrm{ht}(p) > 1$). Therefore $Q = pA[1/b] \cap B$ is a height-one prime divisor of xB (so $Q = P$) and $Q :_B bB = Q$ (since $QB[1/b] \cap B = Q$). Therefore, since xB is $Q = P$-primary, it follows that $xB :_B bB = xB$, so $xB[1/b] \cap B = xB$. Therefore $xB \cap A = (xB[1/b] \cap B) \cap A = xA[1/b] \cap A = xA :_A b^nA$ for all large integers n, and $xB \cap A$ is $p = P \cap A$-primary, hence $xA :_A b^nA$ is p-primary for all large integers n. Also, since $xA \subseteq xA :_A b^nA \subseteq xB$, it follows that $xB = (xA :_A b^nA)B$ for all nonnegative integers n. Further, $xB \subseteq pB \subseteq P$ and $P = \mathrm{Rad}(xB)$, so $P = \mathrm{Rad}(pB)$. However, p is principal, so it follows that pB is P-primary. Then $pB :_B bB = pB$, since $P :_B bB = P$, hence $p \not\subseteq \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}$ and $pB = P$ (by Proposition 3.6), so (2) holds. □

Remark 3.10. If $x \in J$ $(x \neq 0)$ and if xB is P-primary, then J is $P \cap A$-primary and bB is P-primary.

Proof. It is shown in Proposition 2.2 that qB is a height-one prime ideal for each $q \in \mathrm{Ass}_A(A/J)$ and that J is q-primary if and only if bB is qB-primary. The conclusions clearly follow from this. □

We next consider a height-one prime ideal p in A such that $b \notin p$. We want to determine when $pA[1/b] \cap B$ has a principal primary ideal. It is shown in Proposition

3.6 that if $p \not\subseteq \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}$, then $pA[1/b] \cap B = pB$ is a (principal) prime ideal. Therefore in Proposition 3.11 we restrict attention to the case where $p \subseteq \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}$ (and $b \notin p$). In this situation, Proposition 3.11 characterizes when $pA[1/b] \cap B$ has a principal primary ideal.

Proposition 3.11. *Let $p = \pi A$ be a height-one prime ideal in A, assume that $b \notin p \subseteq \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}$, and let $\mathrm{Ass}_A(A/J) = \{q_1, \ldots, q_n\}$. Then the following are equivalent:*
(1) $P = pA[1/b] \cap B$ has a principal primary ideal.
(2) There exist positive integers e, h and nonnegative integers $n_1, \ldots, n_d$ such that $\pi^e {b_1}^{n_1} \cdots {b_d}^{n_d} \in J^h - (bA \cup q_1 J^h \cdots \cup q_n J^h)$, and then $((\pi^e {b_1}^{n_1} \cdots {b_d}^{n_d})/b^h)B$ is P-primary.
(3) There exist a positive integer h and an element $x \in (p \cap J^h)$ such that $(x/b^h)B$ is P-primary.

Proof. (In (2), it should be noted that $n_i < a_i$ for at least one $i \in \{1, \ldots, d\}$, since $\pi^e {b_1}^{n_1} \cdots {b_d}^{n_d} \notin bA$.)

It is clear that (2) $\Rightarrow$ (3) $\Rightarrow$ (1), so it suffices to show that (1) $\Rightarrow$ (2).

For this, assume that (1) holds, say βB is P-primary. Then since $P \cap A = p$ has height one, it follows from Remark 3.10 that $\beta \notin J$. Therefore by Lemma 3.2 let h be the positive integer and x the element in $J^h - (J^{h+1} \cup bA)$ such that $\beta = x/b^h$. Then $x = b^h \beta \in P \cap A = \pi A$, so there exists a positive integer e such that $x \in \pi^e A - \pi^{e+1} A$.

Now $bB = JB$ and $\mathrm{Ass}_B(B/bB) = \{q_i B \mid i = 1, \ldots, n\}$, by Proposition 2.2. Also, B is Cohen-Macaulay, by Proposition 2.2, so every prime divisor of bB has height one. Therefore it follows (since βB is P-primary and $b \notin p = P \cap A$) that b, β is either a B-sequence or $(b, \beta)B = B$. Therefore Proposition 3.3(2) $\Leftrightarrow$ (5) shows that there exists a nonnegative integer m such that $\beta = z/b^m$ with $z \in J^m - (bA \cup q_1 J^m \cup \cdots \cup q_n J^m)$, and by the uniqueness of h and x in the preceding paragraph, it follows that $m = h$ and $z = x$. Then $\pi \in P = \mathrm{Rad}(\beta B)$, so there exists a positive integer n such that $\pi^n = \beta\gamma$ for some $\gamma \in B$.

If $\gamma \notin J$, then by Lemma 3.2 let k be the nonnegative integer and y the element in $J^k - (J^{k+1} \cup bA)$ such that $\gamma = y/b^k$. Then $\pi^n b^{h+k} = xy$, so it follows, using $x \in \pi^e A - \pi^{e+1} A$ and unique factorization in A, that $x = \omega \pi^e {b_1}^{e_1} \cdots {b_d}^{e_d}$ for some unit ω of A and nonnegative integers $e_1, \ldots, e_d$ (and since $b \notin xA$ it follows that at least one $e_i < a_i$).

If $\gamma \in J$, then $\pi^n b^h = x\gamma$, so an argument similar to that in the preceding paragraph yields the same conclusion. $\square$

The next result characterizes when $pA[1/b] \cap B$ is a principal prime ideal (with $p \subseteq \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}$).

Corollary 3.12. *Let $p = \pi A$ be a height-one prime ideal in A such that $b \notin p \subseteq \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}$. Then $P = pA[1/b] \cap B$ is a principal prime ideal if and only if e may be chosen to be 1 in Proposition 3.11(2).*

Proof. Assume first that $pA[1/b] \cap B$ has a principal primary ideal. Then Proposition 3.11 shows that $(\pi^e {b_1}^{n_1} \cdots {b_d}^{n_d}/b^h)B$ is P-primary for some positive integers e, h and nonnegative integers $n_1, \ldots, n_d$. Also, $((\pi^e {b_1}^{n_1} \cdots {b_d}^{n_d}/b^h)B)B[1/b] = \pi^e B[1/b]$, and $((\pi^e {b_1}^{n_1} \cdots {b_d}^{n_d}/b^h)B)B[1/b] \cap B = (\pi^e {b_1}^{n_1} \cdots {b_d}^{n_d}/b^h)B$ (since $(\pi^e {b_1}^{n_1} \cdots {b_d}^{n_d}/b^h)B$ is P -primary and $b \notin P$). Therefore, if $e = 1$, then $\pi^e B[1/b] = pB[1/b]$ is prime, so $(\pi^e {b_1}^{n_1} \cdots {b_d}^{n_d}/b^h)B = P$.

Conversely, if $P = \beta B$, then $\beta \notin J$ (since, otherwise, $P = \mathrm{Rad}(bB)$ and $\mathrm{ht}(P \cap A) > 1$ by Remark 3.10, and this contradicts $\mathrm{ht}(P \cap A) = 1$). Therefore let $\beta = x/b^h$ as in Lemma 3.2. Then it follows as in the proof of Proposition 3.11 that $x = \pi^e {b_1}^{e_1} \cdots {b_d}^{e_d}$ for some positive integer e and nonnegative integers $e_1, \dots, e_d$. Then $\pi^e A[1/b] = xA[1/b] = \beta B[1/b] = PB[1/b] = pA[1/b] = \pi A[1/b]$, so it follows that $e = 1$. $\square$

Remark 3.13. **(1)** If J is a prime ideal, then Proposition 3.3(5) becomes $x \in J^h - J^{h+1}$. Using this, Proposition 3.6, and Proposition 3.11, it readily follows that if J is a prime ideal, then all height-one prime ideals in B (except possibly those that contain bB) have a principal primary ideal. (It was shown in Theorem 2.4 that, in fact, B is a UFD if J is a prime ideal.)
(2) It follows from Proposition 3.11(1) $\Leftrightarrow$ (3) and Proposition 3.3(5) that if p is a height-one prime ideal in A such that $b \notin p$, and if $P = pA[1/b] \cap B$ has no principal primary ideal, then for each positive integer h it holds that $p \cap J^h \subseteq (\bigcup\{qJ^h \mid q \in \mathrm{Ass}_A(A/J)\} \cup bA)$. If A has the property that an ideal contained in a finite union of ideals is contained in one of them, then this becomes more useful and interesting.

With regard to Remark 3.13(2), a theorem of McAdam in [11] states:

Theorem. *Let $J_1, \dots, J_n$ be (not necessarily distinct) ideals of a ring R and $c_1, \dots, c_n \in R$. If I is an ideal of R such that $I \subseteq \bigcup_{i=1}^n (J_i + c_i)$, and if for each i and each maximal prime P of R/J_i, R/P is infinite, then $I + c_i R \subseteq J_i$ for some i.*

(A good reference for such things is [3].)
We have now handled case (1) of Remark 3.1. Before considering case (2) of Remark 3.1 in Section 4, we summarize our results.

Theorem 3.14. *Let $P \in \mathrm{Spec}(B)$ be the radical of a nonzero principal ideal, let $p = P \cap A$, and assume that $\mathrm{ht}(p) = 1$ (so $b \notin p$). Then exactly one of the following holds:*
(1) *$P = \mathrm{Rad}(\pi B)$ for some prime element $\pi \in A$. (This is true if and only if $p = \pi A$ for some prime element $\pi \in A - \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}$ and $P = pB$.)*
(2) *$P = \mathrm{Rad}(\beta B)$ for some $\beta \in B - A$. (This is true if and only if $p = \pi A \subseteq \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}$, $P = pA[1/b] \cap B$, and then β may be chosen to be $(\pi^e {b_1}^{n_1} \cdots {b_d}^{n_d})/b^h$ as in Proposition 3.11(1) $\Rightarrow$ (2).)*

Proof. (1) follows immediately from Proposition 3.6, and (2) follows from Proposition 3.9 and Proposition 3.11(1) $\Leftrightarrow$ (2). $\square$

Corollary 3.15. *If P is a nonzero principal prime ideal in B and $\mathrm{ht}(P \cap A) = 1$, then either:*
(1) *$P = \pi B$ for some prime element $\pi \in A - \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}$; or,*
(2) *$P = \beta B$ for some $\beta \in B - A$ as in Theorem 3.14(2) with $e = 1$.*

Proof. (1) follows immediately from Proposition 3.6, and (2) follows from Proposition 3.9 and Corollary 3.12. $\square$

4. The case where height$(P \cap A) > 1$

In this section we consider case (2) of Remark 3.1, that is, we are interested in the height-one prime ideals P of B such that $\mathrm{ht}(P \cap A) > 1$. To obtain an easy to apply criterion for when these primes P are either principal or have a principal primary ideal we introduce a restriction on the factors $b_1, \dots, b_d$ of b. However, most of the results in this section do not need this restriction, so we do not introduce it till near the end of this section. (We assume throughout this section that A is a Cohen-Macaulay UFD, and $b, c_1, \dots, c_g$, J, $I = (c_1, \dots, c_g)A$, B, and $b = {b_1}^{a_1} \cdots {b_d}^{a_d}$ are as in the previous sections.)

Proposition 4.1 is a variation of Proposition 2.2; this variation is needed below.

Proposition 4.1. *Assume that $b = {b_1}^{a_1} \cdots {b_d}^{a_d}$, where $b_1, \dots, b_d$ are nonassociate nonunits in A and $a_1, \dots, a_d$ are positive integers. Then for $i = 1, \dots, d$ either:*
(1) *$(b_i, I)A = A$ (and this holds if and only if $b_i B = B$); or,*
(2) *$(b_i, I)B = b_i B$, $b_i B \cap A = (b_i, I)A$, and $B/b_i B \cong (A/(b_i, I))[X_1, \dots, X_g]$, so there exists a one-to-one correspondence between the elements of $\mathrm{Ass}_B(B/b_i B)$ and the elements of $\mathrm{Ass}_A(A/(b_i, I)A)$ given by p ($\in \mathrm{Ass}_A(A/(b_i, I)A)) = P \cap A$ with $P \in \mathrm{Ass}_B(B/b_i B)$ and $P = pB$. Also, each $q \in \mathrm{Ass}_A(A/(b_i, I)A)$ has height $g + 1$. Moreover, (2) holds for at least one $i = 1, \dots, d$.*

Proof. It is clear that $J \subseteq (b_i, I)A$ for $i = 1, \dots, d$. Also, $K \subseteq JA[X_1, \dots, X_g]$ (by the proof of Proposition 2.2), so

$$b_i B = (b_i, K)A[X_1, \dots, X_g]/K = (b_i, I)A[X_1, \dots, X_g]/K,$$

since $b \in b_i A$. Therefore the conclusions follow from Lemma 2.1. □

Concerning Proposition 2.2 and Proposition 4.1, it is easy to give examples where $b = {b_1}^{a_1} \cdots {b_d}^{a_d}$ and $b_1 B = B$ (and $(b_1, I)A = A$), as shown in Example 4.2.

Example 4.2. It is possible that bB is a height-one prime ideal and bA is not a prime ideal. It is also possible that $b_i B = B$ for some $i = 1, \dots, d$.

Proof. Let $A = F[X, Y]$ be a polynomial ring in two algebraically independent elements over a field F, let $b_1 = X + 1$, $b_2 = Y$, and $c = c_1 = X$. Then $b = b_1 b_2 = XY + Y$ is such that $(b, c)A = (X, Y)A$ is a height-two prime ideal, so $(X, Y)B = (b, c)B = bB = (XY + Y)B$ (where $B = A[X/(XY + Y)]$) is a height-one prime ideal (by Proposition 2.2), but bA is not a prime ideal. It follows that either $X + 1$ or Y is a unit in B. Since $X \in Y(X + 1)B = bB$, it follows that $-Y = Y(X) - (YX + Y) \in bB$, so $YB \subseteq Y(X + 1)B$, and the opposite inclusion is clear, so $YB = Y(X + 1)B$ and $X + 1$ is a unit in B. □

Remark 4.3. The behavior exhibited in Example 4.2 where bB is prime and bA fails to be prime is not possible in the case where A is local. For $J = bB \cap A$ is then a prime ideal generated by a regular sequence in a local domain and it follows that each subset of the regular sequence generates a prime ideal [13, Theorem 4.1].

Remark 4.4. Concerning Proposition 2.2 and Proposition 4.1, let $p \in \mathrm{Spec}(A)$ have height $g + 1$. Then the following are equivalent:
(1) $p \in \mathrm{Ass}_A(A/J)$;
(2) pB is a height-one prime ideal;
(3) $\mathrm{ht}(pB) = 1$;
(4) there exists a height-one prime ideal $P \in B$ such that $P \cap A = p$; and,
(5) $pB \in \mathrm{Ass}_B(B/bB)$ and $p = pB \cap A$.

Proof. (1) ⇒ (2), by Proposition 2.2.
It is clear that (2) ⇒ (3).

To show that (3) $\Rightarrow$ (4), assume $\mathrm{ht}(p) = g + 1$ and $\mathrm{ht}(pB) = 1$. Let P be a height-one prime divisor of pB, so $p \subseteq P \cap A$, so it follows from Lemma 2.9(3) that $P \cap A = p$. Therefore (4) holds, so (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (1) by Lemma 2.9(3).

Finally, it follows immediately from Proposition 2.2 that (1) $\Leftrightarrow$ (5). $\square$

Proposition 4.5. *If p is a height-one prime ideal in A such that $b \in p$, then $pA = b_iA$ for some $i = 1, \ldots, d$. Also, $pB = b_iB$ is a (height-one) prime (resp., primary) ideal (resp., $= B$) if and only if $(b_i, I)A$ is a (height $g+1$) prime (resp., primary) ideal (resp., $= A$).*

Proof. Assume $b \in pA$. Then $pA = b_iA$ for some $i = 1, \ldots, d$, since $b = {b_1}^{a_1} \cdots {b_d}^{a_d}$ and each b_h is a prime element. Then $pB = b_iB$, so it follows from Proposition 4.1 that pB is a height-one prime ideal (resp., primary ideal) (resp., $= B$) if and only if $(b_i, I)A$ is a height $g + 1$ prime ideal (resp., primary ideal) (resp., $= A$). $\square$

The following remark strengthens the conclusion of Proposition 3.9(1).

Remark 4.6. Let x be a nonzero nonunit in A such that xB is a primary ideal, let $P = \mathrm{Rad}(xB)$, let $p = P \cap A$, and assume that $\mathrm{ht}(p) > 1$. Then there exists $i \in \{1, \ldots, d\}$ such that $(b_i, I)A$ and b_iB are primary, $\mathrm{Rad}(xB) = \mathrm{Rad}(b_iB)$, and $\mathrm{Rad}(xB) \cap A \in \mathrm{Ass}_A(A/J)$.

Proof. Proposition 3.9(1) shows that $p \in \mathrm{Ass}_A(A/J)$ and that $b \in \mathrm{Rad}(xA)$. Further, $x \in p$, so p contains some prime divisor q of xA, and then $b \in \mathrm{Rad}(xA) \subseteq q$, hence $q = b_iA$ for some $i = 1, \ldots, d$ by Proposition 4.5. Therefore $xB \subseteq b_iB \subseteq pB \subseteq P$, so $P = \mathrm{Rad}(b_iB)$ (since $P = \mathrm{Rad}(xB)$). Since B is Cohen-Macaulay, it follows that b_iB is P-primary, so $(b_i, I)A$ is p-primary, by Proposition 4.1. $\square$

We need one more result concerning the height-one prime ideals P in B that have a principal primary ideal. For this result, we introduce for the first time a restriction on the factors $b_1, \ldots, b_d$. This restriction is used to sharpen the conclusion of the next result and its corollaries.

Definition 4.7. (1) With the fixed notation, it is said that $b_1, \ldots, b_d$ satisfy the **Radical Property** with respect to $I = (c_1, \ldots, c_g)A$ in case for each $i \in \{1, \ldots, d\}$ it holds that $b_1 \cdots b_{i-1}b_{i+1} \cdots b_d \notin \mathrm{Rad}((b_i, I)A)$.
(2) A **product-quotient** of elements $x_1, \ldots, x_m$ in A is a product ${x_1}^{n_1} \cdots {x_d}^{n_d}$, where at least one $n_i > 0$ and possibly some n_j are nonpositive.

For an example of Definition 4.7(1), if $B_1, \ldots, B_d, C_1, \ldots, C_g$ are algebraically independent over a field F, then $B_1, \ldots, B_d$ satisfy the Radical Property with respect to $I = (C_1, \ldots, C_g)A$, where $A = F[B_1, \ldots, B_d, C_1, \ldots, C_g]$. Also, $X - 1, \ldots, X - d$ satisfy the Radical Property with respect to $I = YF[X, Y]$, but $X, X + Y$ do not satisfy the Radical Property with respect to I.

And, for an example of Definition 4.7(2), b_1, ${b_2}^5$, and ${b_4}^2/{b_2}^3$ are product-quotients of $b_1, \ldots, b_d$ (with $d \geq 4$), but $1/b_1$ is not.

Proposition 4.8. *Let P be a height-one prime ideal in B, let $p = P \cap A$, and assume that $\mathrm{ht}(p) > 1$ and that $P = \mathrm{Rad}(\beta B)$ for some $\beta \in B$. Then $P = \mathrm{Rad}(\alpha B)$ for some product-quotient α of $b_1, \ldots, b_d$. Moreover, if $b_1, \ldots, b_d$ satisfy the Radical Property with respect to I, then $P = \mathrm{Rad}(b_iB)$ for some $i = 1, \ldots, d$ and $(b_i, I)A$ is a primary ideal.*

Proof. If $\mathrm{ht}(p) > 1$, then $b \in p$, $J \subseteq P \cap A$, and $P = pB$, by Lemma 2.9(3).

Since $b \in P = \mathrm{Rad}(\beta B)$, it follows that there exists a positive integer m such that $b^m = \beta\gamma$ for some $\gamma \in B$. If $\beta \in A$, then since $\mathrm{ht}(p) > 1$, it follows from Proposition 3.9(1) and Remark 4.6 that $P = \mathrm{Rad}(b_i B)$ for some $i = 1, \dots, d$, as desired.

Therefore assume that $\beta \notin A$. We now consider the two cases: (1) $\gamma \in A$; and, (2) $\gamma \notin A$. For both cases, by Lemma 3.2 let $\beta = x/b^h$ (with $x \in J^h - (J^{h+1} \cup bA)$ and $h > 0$).

For case (1), $b^{m+h} = {b_1}^{a_1(m+h)} \cdots {b_d}^{a_d(m+h)} = x\gamma$ in A. If $\gamma A = A$, then it follows that $\beta = x/b^h = (b^{m+h}\gamma^{-1})/b^h = b^m\gamma^{-1} \in A$, and this is a contradiction. Therefore γ is a nonunit in A, so $x = \omega {b_1}^{e_1} \cdots {b_d}^{e_d}$ for some unit ω in A and some nonnegative integers $e_1 \leq a_1(m+h), \dots, e_d \leq a_d(m+h)$ with at least one $e_i > 0$ and at least one $e_j < a_j(m+h)$ (since γ is a nonunit). Therefore $\beta = x/b^h = \omega {b_1}^{e_1 - ha_1} \cdots {b_d}^{e_d - ha_d}$, and at least one $e_i - ha_i > 0$ (since $\beta \in P \neq B$) and at least one $e_j - ha_j < 0$ (since $\beta \notin A$). Reorder the subscripts so that $\beta = u/v$, where $u = \omega {b_1}^{e_1 - ha_1} \cdots {b_i}^{e_d - ha_i}$, $v = {b_{i+1}}^{e_{i+1} - ha_{i+1}} \cdots {b_{i+j}}^{e_{i+j} - ha_{i+j}}$ (for some i, j such that $1 \leq i < j \leq d$) and the exponents in u and v are all positive. If $b_f B = B$ for $f = i+1, \dots, j$, then $\beta B = (u/v)B = uB = ({b_1}^{e_1 - ha_1} \cdots {b_i}^{e_d - ha_i})B$, so it follows that $P = \mathrm{Rad}(b_m B)$ for some $m = 1, \dots, i$, as desired.

Therefore it may be assumed that at least one $b_f B \neq B$ (with $f \in \{i+1, \dots, j\}$), say $f = i+1$. Then $P = \mathrm{Rad}(\alpha B)$ for the product-quotient $\alpha = u/v$ of $b_1, \dots, b_d$. Also, it follows that $u = v(u/v) = v\beta \in vB \cap A \subseteq {b_{i+1}}^{e_{i+1} - ha_{i+1}} B \cap A \subseteq b_{i+1} B \cap A$ $= (b_{i+1}, I)A$, by Proposition 4.1. Therefore $u \in (b_{i+1}, I)A$, so

$$b_1 \cdots b_i \in \mathrm{Rad}((b_{i+1}, I)A).$$

It follows that if $b_1, \dots, b_d$ satisfy the Radical Property with respect to I, then the assumption at the start of this paragraph leads to a contradiction, hence $P = \mathrm{Rad}(b_m B)$ for some $m = 1, \dots, i$ (and then $(b_m, I)A$ is primary by Proposition 4.1).

For case (2), by Lemma 3.2 let $\gamma = y/b^k$ with $k > 0$ and $y \in J^k - (J^{k+1} \cup bA)$. Then $b^{m+h+k} = xy$ in A, so an exactly similar argument to that for case (1) yields the same conclusion. □

It is shown in Corollary 5.3 below that if $d = 1$ (that is, b is a power of a prime element in A), then the conclusion in Proposition 4.8 can be sharpened to $P = \mathrm{Rad}(b_1 B)$ and $(b_1, I)A$ is a primary ideal.

The next corollary shows that if $b_1, \dots, b_d$ satisfy the Radical Property with respect to I, then for each $q \in \mathrm{Ass}_A(A/J)$, it is apparent from q itself whether or not qB has a principal primary ideal.

Corollary 4.9. *Assume that $b_1, \dots, b_d$ satisfy the Radical Property with respect to I. Then the following are equivalent for each $q \in \mathrm{Ass}_A(A/J)$:*
(1) *qB has a principal primary ideal.*
(2) *$(b_i, I)A$ is q-primary for some $i = 1, \dots, d$.*
(3) *$b_i B$ is qB-primary for some $i = 1, \dots, d$.*

Proof. Proposition 4.8 shows that (1) ⇒ (3). (3) ⇔ (2) by Proposition 4.5, and it is clear that (3) ⇒ (1). □

Corollary 4.10. *Assume that $b_1, \dots, b_d$ satisfy the Radical Property with respect to I, let P be a height-one prime ideal in B, and assume that* $\mathrm{ht}(P \cap A) > 1$. *Then the following are equivalent:*

(1) *P is a principal ideal.*
(2) *$(b_i, I)A = P \cap A$ is a prime ideal for some $i = 1, \ldots, d$.*
(3) *$P = b_iB$ for some $i = 1, \ldots, d$.*

Proof. This follows from Proposition 4.8 and Proposition 4.5 as in the proof of Corollary 4.9. $\square$

In Proposition 4.11 we consider the prime divisors of πB, where π is a prime element in A. Since Proposition 3.6 shows that πB is prime if $\pi \notin \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}$, in Proposition 4.11 we restrict attention to the case where $\pi \in \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}$. However, by Proposition 4.1, $(b_i, I)A$ is a primary ideal if and only if b_iB is a primary ideal, so we further restrict attention to the case where $\pi A \notin \{b_1A, \ldots, b_mA\}$ (where $b_1, \ldots, b_m$ are as in Proposition 4.11).

Proposition 4.11. *Let $\mathrm{Ass}_A(A/J) = \{q_1, \ldots, q_n\}$ and let m be the nonnegative integer such that $(b_i, I)A$ is a primary ideal for $i = 1, \ldots, m$ and $(b_i, I)A$ is not a primary ideal for $i = m+1, \ldots, d$. Let $\mathbf{W}_0 = \{q \in \mathrm{Ass}_A(A/J) \mid q = \mathrm{Rad}((b_i, I)A)$ for some $i = 1, \ldots, m\} = \{q_1, \ldots, q_r\}$ (so $0 \leq r \leq m$, since $\mathrm{Rad}((b_i, I)A) = \mathrm{Rad}((b_j, I)A)$ may hold for $i \neq j$ in $\{1, \ldots, m\}$). Let $\pi \in \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}$ be a prime element such that $\pi A \notin \{b_1A, \ldots, b_mA\}$. Assume that π is in exactly s ($0 \leq s \leq r$) of the elements in $\mathbf{W}_0$ and exactly k of the elements in $\mathrm{Ass}_A(A/J) - \mathbf{W}_0$. Then:*
(1) *πB has exactly $s + k$ prime divisors if and only if $\pi A \in \{b_{m+1}A, \ldots, b_dA\}$ and at least s of them have a principal primary ideal. (If $b_1, \ldots, b_d$ satisfy the Radical Property with respect to I, then exactly s of the prime divisors of πB have a principal primary ideal.)*
(2) *Assume that $\pi A \notin \{b_{m+1}A, \ldots, b_dA\}$. Then πB has exactly $s + k + 1$ prime divisors and at least s of them have a principal primary ideal, and at least $s + 1$ of them have a principal primary ideal if there exist h and x as in Proposition 3.11(3) for $p = \pi A$.*

Proof. If $\pi \in q \in \mathrm{Ass}_A(A/J)$, then $\pi B \subseteq qB$ and qB is a height-one prime ideal such that $qB \cap A = q$, by Proposition 2.2, and qB has a principal primary ideal if $\pi A = b_iA$ for some $i = 1, \ldots, m$ (by hypothesis), so πB has at least $s + k$ prime divisors and at least s of them have a principal primary ideal. (If $b_1, \ldots, b_d$ satisfy the Radical Property with respect to I, then only the ideals q_iB (with $i \in \{1, \ldots, m\}$) have a principal primary ideal, by Proposition 4.8.)

Also, if P is a prime divisor of πB, then $\mathrm{ht}(P) = 1$ (since B is Cohen-Macaulay, by Proposition 2.2), so either: $P \cap A = \pi A$ (so $P = \pi A[1/b] \cap B$ and $\pi A \notin \{b_1A, \ldots, b_dA\}$, by Remark 3.8, and $\pi A[1/b] \cap B$ has a principal primary ideal if and only if there exist h and x as in Proposition 3.11(3)); or, $\mathrm{ht}(P \cap A) = g + 1$ (so $b \in P \cap A$, and $P \cap A \in \mathrm{Ass}_A(A/J)$, by Lemma 2.9(3), so P is one of the prime ideals considered in the preceding paragraph).

(1) and (2) readily follow from the preceding two paragraphs. $\square$

We close this section by briefly considering the prime factors of the elements $c_1, \ldots, c_g$.

Remark 4.12. With the fixed notation, the following hold:
(1) if $(b_i, c_j)A$ is a q-primary ideal for some $i = 1, \ldots, d$ and $j = 1, \ldots, g$ (and $\mathrm{ht}(q) = 2$), then $(b_i, c_{j,k})A$ is a Q-primary ideal for at least one prime factor $c_{j,k}$ of c_j.

(2) if $(b_i, c_{j,k})A$ is a q-primary ideal, then it need not be true that $(b_i, c_j)A$ is q-primary, and it then follows from Proposition 4.1 (with $g = 1$) that $(b_i, c_{j,k})B$ is a qB-primary ideal, and Proposition 2.2 shows that $(b, c_j)B$ is not qB-primary.

Proof. Assume that q is a height-two prime ideal in A and that $(b_i, c_j)A$ is q-primary. To prove (1), let $c_{j,1}, \ldots, c_{j,m}$ be (not necessarily distinct) prime elements in A such that $c_j = c_{j,1} \cdots c_{j,m}$ and for $h = 1, \ldots, m$ let $C_h = c_{j,1} \cdots c_{j,h-1} c_{j,h+1} \cdots c_{j,m}A$ and $L_h = (b_i, c_j)A :_A C_h$. Then $L_h = (b_i, c_{j,h})A$, since $(D + xE) :_A xA = (D :_A xA) + E$ holds for all ideals D, E and elements x in a ring A. Therefore each L_h is either q-primary or the ring A. And it is clear that q contains at least one of the ideals $(b_i, c_{j,h})A = L_h$, so L_h is q-primary.

For (2), let $A = F[X, Y, Z]$, $c_j = YZ$, and $b_i = X$; then $(X, Y)A$ is $(X, Y)A$-primary, but $(X, YZ)A$ is not $(X, Y)A$-primary. □

5. The case where b is a primary element

In this section we give some additional results, under the assumption that b is a power of a prime element in A. We say that b with this property is a **primary element**.

Theorem 5.1 shows that if b is a primary element and J is not a primary ideal, then B is not a UFD.

Theorem 5.1. *Assume that $b = {b_1}^{a_1}$ is a power of a prime element b_1 in A. If P is a height-one prime ideal in B that is the radical of a principal ideal, if* $\mathrm{ht}(P \cap A)$ $= 1$*, and if $P \cap A \subseteq q \in \mathrm{Ass}_A(A/J)$, then $P \cap A \subseteq \mathrm{Rad}(J)$. Therefore, if J is not a primary ideal, then for each height-one prime ideal p in A that is contained in at least one, but not all, prime divisors of J it holds that $pA[1/b] \cap B$ has no principal primary ideals.*

Proof. The last statement clearly follows from the first, so it suffices to show that $P \cap A \subseteq \mathrm{Rad}(J)$.

For this, suppose that $q \neq q'$ are prime divisors of J such that $P \cap A \subseteq q$ and $P \cap A \not\subseteq q'$. (It is shown in Proposition 2.2 that $\mathrm{ht}(q) = g + 1 = \mathrm{ht}(q')$, so $P \cap A$ is a proper subset of q and $b \notin P \cap A$, by Lemma 2.9(3).) Let $p = P \cap A = \pi A$, so $b \notin \pi A$, and $P = pA[1/b] \cap B$, so P is the only prime ideal in B that lies over p (by Lemma 2.9(2)). We now show that P does not have a principal primary ideal (so this contradiction to the hypothesis shows that J is primary).

For this, assume that $P = \mathrm{Rad}(\beta B)$. Then $\beta \notin p = P \cap A$, since qB is a height-one prime divisor of aB for each nonzero $a \in p$ (by Proposition 2.2). Therefore let $\beta = x/b^h$, where $x \in J^h - (J^{h+1} \cup bA)$ (and $h > 0$). Then $\pi^n \in \beta B$, since $P = \mathrm{Rad}(\beta B)$ and $\pi \in P \cap A = p$. Therefore $\pi^n = \beta\gamma$ for some $\gamma \in B$, and we now consider the two cases: (1) $\gamma \in A$; and, (2) $\gamma \in B - A$.

For case (1), it follows that $\pi^n b^h = x\gamma$. Since $x \notin bA$ and bA is primary, it follows from unique factorization in A that $x = \omega {b_1}^f \pi^e$ for some unit ω of A, for some nonnegative integer $f < a_1$, and for some positive integer $e \leq n$ ($e > 0$, since $x = \beta b^h \in \beta B \cap A \subseteq P \cap A = \pi A$). If $f = 0$, then $\pi^e A = xA \subseteq J$, hence $p \subseteq J \subseteq q'$, and this is a contradiction. Therefore $f > 0$, so $\beta = x/b^h = (\omega {b_1}^f \pi^e)/b^h$ $= \omega\pi^e/{b_1}^{ha_1-f}$, so $\pi^e A \subseteq {b_1}^{ha_1-f}B \cap A$. Now $\mathrm{Rad}({b_1}^{ha_1-f}B) = \mathrm{Rad}(b_1 B) = \mathrm{Rad}(bB)$, and $\mathrm{Rad}(bB) \cap A = \mathrm{Rad}(J)$, by Proposition 2.2. Therefore $\pi^e \in \mathrm{Rad}(J)$, so $\pi \in q'$, and this contradicts the choice of π. Therefore (1) does not hold, so (2) must hold.

Since (2) holds, let $\gamma = y/b^k$, where $y \in J^k - (J^{k+1} \cup bA)$ (and $k > 0$). Then it follows that $\pi^n b^{h+k} = xy$ in A, and $h + k \geq 2$. Therefore the unique factorization in A shows that $xy \in b^{h+k}A = {b_1}^{a_1(h+k)}A$, and this contradicts the fact that $x \notin {b_1}^{a_1}A$ and $y \notin {b_1}^{a_1}A$. Therefore (2) also does not hold, so the supposition that J has at least two distinct prime divisors yields the contradiction that P is not the radical of a principal ideal. □

In the next theorem we show (under the assumption that A is a UFD and b is a primary element) that the converse of Theorem 2.4(3) holds; that is, we characterize when B is a Krull domain with torsion class group in this case.

Theorem 5.2. *Assume that $b = {b_1}^{a_1}$ is the power of a prime element b_1 in A. Then the following are equivalent:*
(1) *B is a Krull domain with finite cyclic class group.*
(2) *B is a Krull domain with torsion class group.*
(3) *J is primary and integrally closed.*

Proof. That (1) implies (2) is clear, and (2) implies (3) by Remark 2.3 and Theorem 5.1. Finally, (3) implies (1) by Theorem 2.4(2). □

Our next result is the promised corollary of Proposition 4.8.

Corollary 5.3. *Assume that b is a primary element, so $d = 1$, and that P is a height-one prime ideal in B such that* $\mathrm{ht}(P \cap A) > 1$. *If P has a principal primary ideal, then $P = (P \cap A)B = \mathrm{Rad}(b_1B)$ and J is a primary ideal.*

Proof. It is clear that b_1 satisfies the Radical Property with respect to I, so the conclusion follows from Proposition 4.8. □

The next result shows that the necessary and sufficient condition in Proposition 3.11(2) for $pA[1/b] \cap B$ to have a principal primary ideal (where $b \notin p \subseteq \bigcup\{q \mid q \in \mathrm{Ass}_A(A/J)\}$) is easier to use if b is a primary element.

Remark 5.4. Let $b = {b_1}^{a_1}$ be a power of a prime element in A, let $\mathrm{Ass}_A(A/J) = \{q_1, \ldots, q_n\}$, let $b_1 \notin p = \pi A \subseteq q_1 \cup \ldots \cup q_n$, and let $P = pA[1/b] \cap B$. Then it follows from Proposition 3.11 that $p = \pi A$ is such that P has a principal primary ideal if and only if there exist positive integers e, h and a nonnegative integer n_1 such that $\pi^e {b_1}^{n_1} \in J^h - (bA \cup q_1 J^h \cdots \cup q_n J^h)$ (so $n_1 < a_1$), and then $((\pi^e {b_1}^{n_1})/b^h)B$ is P-primary. If $b = b_1$ is a prime element in A, then $p = \pi A$ ($\neq bA$) is such that P has a principal primary ideal if and only if there exist positive integers e, h such that $\pi^e \in J^h - (bA \cup q_1 J^h \cdots \cup q_n J^h)$, and then $(\pi^e/b^h)B$ is P-primary.

We close this section with the following comment that extends the usefulness of the preceding results.

Remark 5.5. **(1)** If some c_j is a power of a prime element, then the results in this section hold for the ring $A[J/c_j]$.
(2) If $b_iB = B$ for all but one i (say $b_1B \neq B$), then the results in this section hold concerning B – since $1/(b_2 \cdots b_d)$ is a unit in B and $C = A[1/(b_2 \cdots b_d)]$ is a UFD such that $bC = {b_1}^{a_1}C$ is a power of a prime element in C, $b, c_1, \ldots, c_g$ is a C-sequence, and $B = C[J/b] = C[J/{b_1}^{a_1}]$.
(3) The results in this section hold for the Rees ring $\mathbf{R}(A, J)$, as is shown in the next section.

6. Application to the Rees ring

In this section we apply the previous results to the Rees ring $\mathbf{R}(A, J)$, where A and J are as in the previous sections.

Remark 6.1. Let A be a Cohen-Macaulay UFD, let J be generated by the A-sequence $b, c_1, \ldots, c_g$, and let $A^* = A[u]$, where u is an indeterminate. Then A^* is a Cohen-Macaulay UFD, $u, b, c_1, \ldots, c_g$ is an A^*-sequence, and $\mathbf{R}(A, J) = A^*[b/u,\ c_1/u,\ \ldots,\ c_g/u] = A[u, tb, tc_1, \ldots, tc_g]$. Therefore the results in the previous sections apply with A^* and $u, b, c_1, \ldots, c_g$ in place of A and $b, c_1, \ldots, c_g$. And u is a prime element in A^*, so, in particular, the results in Section 5 apply to $\mathbf{R} = \mathbf{R}(A, J)$.

Our first result is, essentially, a restatement of Theorem 2.4 in terms of A^* and $(u, J)A^* = (u, b, c_1, \ldots, c_g)A^*$.

Proposition 6.2. *Assume that A is an integrally closed Cohen-Macaulay domain and that J is generated by the A-sequence $b, c_1, \ldots, c_g$. Let $\mathbf{R} = \mathbf{R}(A, J)$ and let $A^* = A[u]$. Then:*
(1) *If J is integrally closed, then $\mathbf{R}$ is integrally closed and there is a surjective homomorphism $\varphi : \mathrm{Cl}(\mathbf{R}) \to \mathrm{Cl}(\mathbf{R}[1/u])$ $(= \mathrm{Cl}(A[u,t]) = \mathrm{Cl}(A))$ whose kernel is generated by the classes of the elements of $\mathrm{Ass}_{\mathbf{R}}(\mathbf{R}/u\mathbf{R})$.*
(2) *If J is integrally closed and primary, and if $\mathrm{Cl}(A)$ is torsion (resp., finite) (resp., trivial), then $\mathrm{Cl}(\mathbf{R})$ is torsion (resp., finite) (resp., finite cyclic).*
(3) *If J is a prime ideal, then $u\mathbf{R} \in \mathrm{Spec}(\mathbf{R})$ and the divisor class groups $\mathrm{Cl}(A)$ and $\mathrm{Cl}(\mathbf{R})$ are isomorphic.*

Proof. For (1), if J is integrally closed and is generated by an A-sequence, then so is $(u, J)A^*$, so (1) follows from Theorem 2.4(1) (with $\mathbf{R}$ and u in place of B and b).

For (2), if J is integrally closed and primary, then so is $(u, J)A^*$, so (2) follows from Theorem 2.4(2).

For (3), if J is prime, then so is $(u, J)A^*$ and uA^* is also prime, so (3) follows from Theorem 2.4(3). □

The next result is used in Theorem 6.4 to characterize when $\mathbf{R}(A, J)$ is a Cohen-Macaulay UFD.

Proposition 6.3. *If P is a height-one prime ideal in $\mathbf{R}(A, J)$ that is the radical of a principal ideal, if $\mathrm{ht}(P \cap A[u]) = 1$, and if $P \cap A[u] \subseteq qA[u]$, for some $q \in \mathrm{Ass}_A(A/J)$, then $P \cap A[u] \subseteq (\mathrm{Rad}(J))A[u]$. Therefore, if J is not a primary ideal, then for each height-one prime ideal p in $A[u]$ that is contained in at least one, but not all, prime divisors of $(u, J)A[u]$ it holds that $pA[u, t] \cap \mathbf{R}(A, J)$ has no principal primary ideals.*

Proof. Note that u is a prime element in A^*, so the conclusion follows from Theorem 5.1. □

It follows immediately from Proposition 6.3 that $\mathbf{R}$ is not a UFD if J is not a primary ideal. In the next result we characterize when $\mathbf{R}$ is a Cohen-Macaulay UFD.

Theorem 6.4. *$\mathbf{R}(A, J)$ is a Cohen-Macaulay UFD if and only if J is prime, and then $u\mathbf{R}(A, J)$ is a prime ideal.*

Proof. If J is prime, then so is $(u, J)A^*$, so $\mathbf{R}(A, J)$ is a UFD and $u\mathbf{R}(A, J)$ is a prime ideal, by Proposition 6.2(3).

For the converse, assume that J is not a prime ideal. By Proposition 6.3, if J is not primary, then $\mathbf{R}(A, J)$ is not a UFD, so it may be assumed that J is primary, say $\mathrm{Rad}(J) = q$. Then $u\mathbf{R}(A, J)$ is primary for $(u, q)\mathbf{R}(A, J)$, by Proposition 2.2, and u is part of a minimal basis for $(u, q)\mathbf{R}(A, J)$ (since all elements of negative degree are a multiple of u), so it follows that $(u, q)\mathbf{R}(A, J)$ is not a principal prime ideal, hence $\mathbf{R}(A, J)$ is not a UFD. □

The following corollary strengthens Corollary 2.7 by removing the "permutable" hypothesis.

Corollary 6.5. *If J is prime, then each of the rings $B_j = A[J/c_j]$ is a Cohen-Macaulay UFD and $c_jB_j \in \mathrm{Spec}(B_j)$.*

Proof. If J is prime, then $\mathbf{R}(A, J)$ is a UFD, by Theorem 6.4, so each ring $\mathbf{B}_j = \mathbf{R}(A, J)[1/(tc_j)]$ is a UFD. However, $\mathbf{B}_j = B_j[tc_j, 1/(tc_j)]$ and $tc_jB[tc_j]$ is prime, since tc_j is transcendental over B_j. By Nagata's Theorem it follows that $B_j[tc_j]$ is a UFD, hence B_j is a UFD. □

Theorem 6.6. *The following are equivalent:*
(1) $\mathbf{R}(A, J)$ *is a Krull domain with finite cyclic class group.*
(2) $\mathbf{R}(A, J)$ *is a Krull domain with torsion class group.*
(3) J *is primary and integrally closed.*

Proof. Note that u is a prime element in A^*. Also, J is primary and integrally closed if and only if $(u, J)A^*$ is primary and integrally closed, so the conclusion follows from Theorem 5.2. □

Corollary 6.7. *If the equivalent conditions in Theorem 6.6 hold, then for $j = 1, \ldots, g$ it holds that $B_j = A[J/c_j]$ is a Krull domain with finite cyclic class group.*

Proof. If $\mathbf{R}(A, J)$ is a Krull domain with finite cyclic class group, then so is $\mathbf{B}_j = \mathbf{R}(A, J)[1/(tc_j)] = B_j[tc_j, 1/(tc_j)]$, so it follows much as in the proof of Corollary 6.5 that B_j is a Krull domain with finite cyclic class group. □

Proposition 6.8. *Let $q \in \mathrm{Ass}_A(A/J)$, assume that $(x/b^h)B$ is a qB-primary ideal, and let $\mathbf{R} = \mathbf{R}(A, J)$. If t^hx, tb is an $\mathbf{R}$-sequence, then J is a q-primary ideal and bB is a qB-primary ideal.*

Proof. Let $\mathbf{B} = B[tb, 1/(tb)]$, so $\mathbf{B} = \mathbf{R}[1/(tb)]$ and $u\mathbf{B} = b\mathbf{B}$.

Now $(x/b^h)B$ is qB-primary, by hypothesis, so $t^hx\mathbf{B}$ is $q\mathbf{B}$-primary, so $t^hx\mathbf{B} \cap \mathbf{R} = t^hx\mathbf{R} :_{\mathbf{R}} (tb)^n\mathbf{R}$ is $q\mathbf{B} \cap \mathbf{R} = (u, q)\mathbf{R}$-primary for all large integers n. Therefore, if t^hx, tb is an $\mathbf{R}$-sequence, then $t^hx\mathbf{R}$ is $(u, q)\mathbf{R}$-primary. Since u is a prime element in $A[u]$, it follows from Corollary 5.3 that $(u, J)A[u]$ is $(u, q)A[u]$-primary and that $u\mathbf{R}$ is $(u, q)\mathbf{R}$-primary. Therefore J is q-primary and $u\mathbf{B} = b\mathbf{B}$ is $(u, q)\mathbf{B} = q\mathbf{B}$-primary, so it follows from $\mathbf{B} = B[tb, 1/(tb)]$ that bB is qB-primary. □

Our final result in this section characterizes when $pA[u, t] \cap \mathbf{R}(A, J)$ has a principal primary ideal and when it is a principal prime ideal, where $p \subseteq \bigcup\{(u, q)A[u] \mid q \in \mathrm{Ass}_A(A/J)\}$.

Proposition 6.9. *Let $p = \pi A[u]$ be a height-one prime ideal in $A[u]$ such that $u \notin p \subseteq \bigcup\{(u, q)A[u] \mid q \in \mathrm{Ass}_A(A/J)\}$, let $P = pA[u, t] \cap \mathbf{R}(A, J)$, and let $\mathrm{Ass}_A(A/J) = \{q_1, \ldots, q_n\}$. Then the following hold:*

(1) P has a principal primary ideal if and only if there exist positive integers e, h such that $\pi^e \in (u, J)^h A[u] - (uA[u] \cup (u, q_1)(u, J)^h A[u] \cup \cdots \cup (u, q_n)(u, J)^h A[u])$, and then $(\pi^e t^h)\mathbf{R}(A, J)$ is P -primary.
(2) P is a principal ideal if and only if e in (1) can be chosen to be 1.

Proof. (1) follows immediately from Remark 5.4, and (2) follows from (1) and Corollary 3.12. □

7. An application

We end by expanding on the example mentioned in the Introduction in order to illustrate some of the results in this paper. In particular, it is shown that each finitely generated abelian group is $\mathrm{Cl}(B)$ where B is a monoidal transform $A[J/b]$, with A a Cohen-Macaulay UFD. In the following we assume for simplicity that the ring R contains a field of characteristic zero, although it will be clear that substantially less would suffice.

Proposition 7.1. *Let R be an integrally closed Cohen-Macaulay domain containing a field of characteristic zero, let $A = R[X, Y]$, where X, Y are indeterminates, let $\pi = Y(Y-1)\cdots(Y-h)$, and let $B = A[\pi/X]$. Then B is integrally closed, Cohen-Macaulay and there is a short exact sequence $0 \to \mathbb{Z}^h \to \mathrm{Cl}(B) \to \mathrm{Cl}(R) \to 0$.*

Proof. It follows by Proposition 2.2 that B is Cohen-Macaulay. Let $p_i = (X, Y-i)A$ and $P_i = p_i B$ for $i = 0, 1, \ldots, h$. Then $J = (X, \pi)A$ has reduced primary decomposition $J = p_0 \cap \cdots \cap p_h$, and thus by Lemma 2.1, we have the reduced primary decomposition $XB = P_0 \cap \cdots \cap P_h$. By Theorem 2.4(1), there is a surjective homomorphism $\varphi : \mathrm{Cl}(B) \to \mathrm{Cl}(A[1/X])$ whose kernel is generated by the classes of the members of $\mathrm{Ass}_B(B/XB) = \{P_0, P_1, \ldots, P_h\}$. Also, by Corollary 4.9, the P_i have no principal primary ideals, and thus these height-one prime ideals have infinite order in the class group of B. The primary decomposition of XB produces the relation $0 = \sum_{i=0}^h [P_i]$ in $\mathrm{Cl}(B)$ among the classes $[P_i]$ of the P_i. Thus $\ker(\varphi)$ is a sum of the infinite cyclic groups generated by the classes $[P_i]$, $i = 1, \ldots, h$. Further, the sum $\sum_{i=1}^h [P_i]\mathbb{Z}$ is direct since if we had a relation $0 = \sum_{i=1}^h s_i[P_i]$, $s_i \in \mathbb{Z}$, then after possibly multiplying by a principal ideal this would give $\beta B = \bigcap_{i=1}^h P_i^{(k_i)}$ in B, $\beta \in B$, $0 \le k_i$. But if β is an element in B whose prime divisors are all among $\{P_0, \ldots, P_h\}$, then in $B[1/X] = A[1/X]$, β becomes a unit (since all of its prime divisors extend to $B[1/X]$). However, the units in $A[1/X] = R[X, Y, 1/X]$ are of the form uX^i, u a unit in R and $i \in \mathbb{Z}$. Thus $\beta B = XB = P_0 \cap \cdots \cap P_h$ gives the only relation in $\mathrm{Cl}(B)$ among the $[P_i]$.

Therefore we have a short exact sequence $0 \to \mathbb{Z}^h \to \mathrm{Cl}(B) \to \mathrm{Cl}(A[1/X]) \to 0$. But X is a prime element in A, and thus by Nagata's Theorem [5, Corollary 7.3], the canonical map $\mathrm{Cl}(A) \to \mathrm{Cl}(A[1/X])$ is an isomorphism. Thus since $\mathrm{Cl}(A) \cong \mathrm{Cl}(R)$ [5, Theorem 8.1], we get the short exact sequence $0 \to \mathbb{Z}^h \to \mathrm{Cl}(B) \to \mathrm{Cl}(R) \to 0$. □

Lemma 7.2. *Let A be an integrally closed Cohen-Macaulay domain, let x, $y \in A$ be such that xA and $(x, y)A$ are distinct nonzero prime ideals, and let $B = A[y^k/x]$. Then B is integrally closed and Cohen-Macaulay, and $J = (x, y^k)A$ is p-primary for $p = (x, y)A$. Moreover, $P = pB$ is a prime ideal of B and $xB = (x, y^k)B =$*

JB is the k-th symbolic power $P^{(k)} = P^k B_P \cap B$ of P, and $P^{(j)}$ is not principal for $0 < j < k$.

Proof. The ideal $J = (x, y^k)A$ is integrally closed by [4, Exercise 4.23], and thus by Remark 2.3, B is integrally closed. Also $xB = (x, y^k)B$ by Lemma 2.1, and B is Cohen-Macaulay by Proposition 2.2.

To see that $(x, y^k)B = P^{(k)}$, let W be an indeterminate. We have

$$B = A[y^k/x] \cong A[W]/(xW - y^k) = A[w]$$

where $w = y^k/x$ is the residue class of W [12, (11.13)]. Further $w \notin P = (x, y)A[w]$; for $w \in (x, y)A[w]$ would give $W \in (x, y, xW - y^k)A[W] = (x, y)A[W]$. Thus, since $xw = y^k$, we get $x \in (x, y)^k A[w]_P \cap A[w] = P^{(k)}$. Thus $(x, y^k)A[w] \subseteq (x, y)^k A[w]_P \cap A[w] = P^{(k)}$. The opposite inclusion is clear.

Now suppose $\beta B = P^{(j)}$ for some $\beta \in B$ and $0 < j < k$. Then $JB = (x, y^k)B = P^{(k)}$ is properly contained in βB and we have $x = \beta b$, where b is a nonunit of B. It follows that $b \in B - xB$ and $\beta \in B - xB$, for otherwise we get that b and β are units of B.

Since xA is prime, it is clear that $b, \beta \in A$ is impossible.

If $x = \beta b$, $b \in B - A$, $\beta \in A$, then for some $n \geq 1$ we have $x^n b = a_1 y^k + a_2 x$, with $a_1 \in A - xA$, $a_2 \in A$. Hence $x^n x = x^n b\beta = a_1 y^k \beta + a_2 x\beta$. But then $a_1 y^k \beta \in xA$, a prime ideal, giving the contradiction that $\beta \in xA$.

Similarly, if $x = \beta b$, $\beta \in B - A$, $b \in A$, then for some $n \geq 1$ we have $x^n \beta = a_1 y^k + a_2 x$, with $a_1 \in A - xA$, $a_2 \in A$. Hence $x^n x = x^n \beta b = a_1 y^k b + a_2 x b$, giving the contradiction $a_1 y^k b \in xA$.

The above three paragraphs show $x = \beta b$, $\beta \in B - A$, $b \in B - A$. Thus we have for some $h \geq 1$, $x^h b = b_1 y^k + b_2 x$ for $b_1 \in A - xA$, $b_2 \in A$, and similarly for some $n \geq 1$, $x^n \beta = c_1 y^k + c_2 x$, $c_1 \in A - xA$, $c_2 \in A$. Then $x^{1+n+h} = x^{n+h}\beta b = (x^n \beta)(x^h b) = (c_1 y^k + c_2 x)(b_1 y^k + b_2 x)$, giving the contradiction $c_1 b_1 y^{2k} \in xA$. $\square$

Lemma 7.3. *Let $A \subseteq B$ be Noetherian rings, let p be a prime ideal in A, and let σ be a nonunit in A. Assume that $(p, \sigma)A = A$ and that $B \subseteq A[1/\sigma]$. Then pB is a prime ideal and $\mathrm{ht}(pB) = \mathrm{ht}(p)$.*

Proof. Note first that $pA[1/\sigma] \neq A[1/\sigma]$, since $\sigma \notin p$, so $P = pA[1/\sigma] \cap B$ is a prime ideal such that $pB \subseteq P$, and $\mathrm{ht}(P) = \mathrm{ht}(pB[1/\sigma]) = \mathrm{ht}(pA[1/\sigma]) = \mathrm{ht}(p)$. Therefore it suffices to show that $P = pB$, and this follows if it is shown that P is the only prime divisor of pB (since $(pB)B[1/\sigma] = (pB)A[1/\sigma] = pA[1/\sigma] = PB[1/\sigma]$).

Therefore let Q be a prime divisor of pB. Since $(pB)B[1/\sigma]$ is prime, it follows that either: (1) $Q = P$ (as desired); or, (2) $\sigma \in Q$. However, (2) cannot hold, since if $pB + \sigma B \subseteq Q$, then $A = (p, \sigma)A \subseteq Q$. Therefore (1) holds, so pB is prime and $\mathrm{ht}(pB) = \mathrm{ht}(p)$. $\square$

Proposition 7.4. *Let R, A, B, and h be as in Proposition 7.1 and for $i = 1, \ldots, k$ let $b_i = X - (h + i)$, $c_i = Y - (h + i)$, and $b'_i = b/b_i$, where $b = b_1 b_2 \cdots b_k$. Let $J = (b, b'_1 c_1^{n_1}, \ldots, b'_k c_k^{n_k})B$ and let $C = B[J/b]$. Then C is integrally closed Cohen-Macaulay, and there is a short exact sequence $0 \to \bigoplus_{i=1}^k \mathbb{Z}/n_i\mathbb{Z} \to \mathrm{Cl}(C) \to \mathrm{Cl}(B) \to 0$.*

Proof. The ideals $b_i B$ and $p_i = (b_i, c_i)B$ are prime by Lemma 7.3. We have $J = (b, b'_1 c_1^{n_1}, \ldots, b'_k c_k^{n_k})B = \bigcap_{i=1}^k (b_i, b'_i c_i^{n_i})B$, and $(b_i, b'_i c_i^{n_i})B = (b_i, c_i^{n_i})B$ is p_i-primary for $p_i = (b_i, c_i)B$. Further since any maximal ideal of B can contain at most one of the b_i, it follows that for each maximal ideal M of B, JB_M is either of the principal class or $JB_M = B_M$. Let $P_i = p_iC$. We get by Proposition 2.2, that $b_iC_{B-M} = (b_i, c_i^{n_i})C_{B-M} = P_i^{(n_i)}C_{B-M}$ if $b_i \in M$. Thus C is Cohen-Macaulay and integrally closed by Lemma 7.2. Further, if $b_i \in M$, then $P_i^{(j)}C_{B-M}$ is not principal for $0 < j < n_i$, by Lemma 7.2.

It follows that the class of P_i in $\mathrm{Cl}(C)$ has order n_i. By [5, Corollary 7.2], the canonical homomorphism $\varphi : \mathrm{Cl}(C) \to \mathrm{Cl}(C[1/b]) = \mathrm{Cl}(B[1/b])$ is surjective and the kernel of φ is generated by the classes of height-one primes of C which contain bC. Therefore $\ker(\varphi)$ is generated by the members of $\mathrm{Ass}_C(C/bC) = \{P_1, \dots , P_k\}$, and thus $\ker(\varphi)$ is a sum of cyclic groups $\mathbb{Z}/n_i\mathbb{Z}$, where $\mathbb{Z}/n_i\mathbb{Z}$ is generated by the class $[P_i]$ of $P_i = (b_i, c_i)C$ in $\mathrm{Cl}(C)$. Further, this sum is direct, since if we had a relation $0 = \sum_{i=1}^k s_i[P_i]$, $0 \le s_i < n_i$, this would give $\beta C = \bigcap_{i=1}^n P_i^{(s_i)}$ for some $\beta \in C$. But then passing to the monoidal transform $C[1/b_i']$ of $B[1/b_i']$, we would have $P_i^{(s_i)}B[1/b_i']$ is principal, which, by Lemma 7.2, is not the case. Thus we have a short exact sequence $0 \to \bigoplus_{i=1}^k \mathbb{Z}/n_i\mathbb{Z} \to \mathrm{Cl}(C) \to \mathrm{Cl}(B[1/b]) \to 0$. But b is a product of prime elements in B, and thus by Nagata's Theorem [5, Corollary 7.3], the canonical map $\mathrm{Cl}(B) \to \mathrm{Cl}(B[1/b])$ is an isomorphism. Thus we get the short exact sequence $0 \to \bigoplus_{i=1}^k \mathbb{Z}/n_i\mathbb{Z} \to \mathrm{Cl}(C) \to \mathrm{Cl}(B) \to 0$. ☐

Theorem 7.5. *Let R be an integrally closed Cohen-Macaulay domain containing a field of characteristic zero and let G be a finitely generated abelian group. Then there exists a monoidal transform $C = A[J/b]$ of $A = R[X, Y]$ (where X, Y are indeterminates) such that $0 \to G \to \mathrm{Cl}(C) \to \mathrm{Cl}(R) \to 0$ is an exact sequence.*

Proof. Let $G = \mathbb{Z}^h \oplus (\bigoplus_{i=1}^k \mathbb{Z}/n_i\mathbb{Z})$. By Proposition 7.1, $B = A[\pi/X]$ is an integrally closed Cohen-Macaulay domain such that there exists an exact sequence

$$0 \to \mathbb{Z}^h \to \mathrm{Cl}(B) \xrightarrow{g} \mathrm{Cl}(R) \to 0.$$

Let $J = (b, b_1'c_1^{n_1}, \dots , b_k'c_k^{n_k})B$ where $b_i = X - (h + i)$, $c_i = Y - (h + i)$ for $i = 1, \dots , k$, $b = b_1b_2 \cdots b_k$, $b_i' = b/b_i$. Then by Proposition 7.4, $C = B[J/b]$ is integrally closed Cohen-Macaulay, and there is a short exact sequence

$$0 \to \bigoplus_{i=1}^k \mathbb{Z}/n_i\mathbb{Z} \to \mathrm{Cl}(C) \xrightarrow{f} \mathrm{Cl}(B) \to 0.$$

Thus by the well known exact sequence

$$0 \to \ker(f) \to \ker(gf) \to \ker(g) \to \mathrm{coker}(f),$$

we see that $\ker(gf) = \mathbb{Z}^h \oplus (\bigoplus_{i=1}^k \mathbb{Z}/n_i\mathbb{Z})$. Thus we have the exact sequence

$$0 \to G \to \mathrm{Cl}(C) \xrightarrow{gf} \mathrm{Cl}(R) \to 0,$$

where $C = A[\pi/X, b_1'c_1^{n_1}/b, \dots , b_k'c_k^{n_k}/b] = A[\pi b/bX, b_1'c_1^{n_1}X/bX, \dots , b_k'c_k^{n_k}X/bX]$. ☐

Corollary 7.6. *Each finitely generated abelian group G is $\mathrm{Cl}(B)$, where B is a monoidal transform $A[J/b]$, with A a Cohen-Macaulay UFD.*

Proof. Take R to be a field of characteristic zero in Theorem 7.5. ☐

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